Math 265, Practice Midterm 1

Name: ______________________________

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	16	
2	16	
3	15	
4	15	
5	15	
6	10	
7	13	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) If A^2 makes sense then A is a square matrix.

(b) A system of linear equations can not have exactly 2 solutions.

(c) If $AB = AC$ and $A \neq 0$ then $B = C$.

(d) Let W be a subspace of $\mathbb{R}^n$. If $v \in W$ then $-v \in W$.

(e) $\det(2A) = 2\det(A)$.

(f) A square matrix A is invertible if and only if $AX = b$ has at most one solution.

(g) Let $v_1, \ldots, v_m \in \mathbb{R}^n$. If $m \leq n$ then $v_1, \ldots, v_m$ is always linearly independent.

(h) Let $T : \mathbb{R}^m \to \mathbb{R}^n$ is linear transformation. Then if $v_1, \ldots, v_m$ is a basis of R^m then $T(v_1), \ldots, T(v_m)$ is a basis of $\mathbb{R}^n$.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Answer	T	T	F	T	F	T	F	F

2. Quick Questions, $A,\ B,\ C,\ X,\ b$ are always matrices here:

(a) Suppose that $\det(A) = \det(A^{-1})$. Find $\det(A)$.
Solutions: Since $\det(A) = \det(A)^{-1}$, $\det(A)^2 = 1$. So $\det(A) = \pm 1$.

(b) Suppose $AX = 2X$ and $A^3X = aX$. Then $a =$?
Solutions: $a = 8$

(c) Let

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & -2 & 1 \\ 1 & 2 & -2 \end{bmatrix}.$$

Compute $2I_2 - AB^T$.

Solutions: $2I_2 - AB^T = \begin{bmatrix} 3 & -3 \\ -7 & 8 \end{bmatrix}$

(d) Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} |ab = 0 \right\}$. Is V a subspace of $\mathbb{R}^2$? why?

Solutions: No. V is not closed under addition. For example, $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are in V. But $u + v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in V.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Suppose that

$$T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}1\\3\end{bmatrix}, \; T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}3\\1\end{bmatrix}.$$

(a) Find the standard matrix for T.

We easily find that $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}$. So

$$T(\begin{bmatrix}1\\0\end{bmatrix}) = \frac{1}{2}T(\begin{bmatrix}1\\1\end{bmatrix}) + \frac{1}{2}T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}2\\2\end{bmatrix}.$$

and

$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{1}{2}T(\begin{bmatrix}1\\1\end{bmatrix}) - \frac{1}{2}T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}-1\\1\end{bmatrix}.$$

Therefore the standard matrix of T is given by

$$(T(e_1), T(e_2)) = \begin{bmatrix}2 & -1\\2 & 1\end{bmatrix}$$

(b) Is T onto? Explain.

Solutions: The standard matrix of T is invertible because it has determinate $4 \neq 0$. So T is onto and one-to-one.

(c) Is T one-to-one? Explain.

4. Consider the following linear system

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & a^2-4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \\ a \end{pmatrix}$$

(a) Determine all values of a such that the system has no solution.

Solutions: The augmented matrix of the above system is

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & 2 & 1 & 7 \\ 1 & 1 & a^2-4 & a \end{pmatrix}.$$

Then we get the echelon form:

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & a^2-4 & a-2 \end{pmatrix}.$$

Then the system has no solution if and only if $a^2 - 4 = 0$ and $a - 2 \neq 0$. So $a = -2$

(b) Determine all values of a such that the system has infinitely many solutions.

Solutions: The system has infinitely many solutions if and only if $a^2 - 4 = a - 2 = 0$. So $a = 2$.

(c) Determine all values of a such that the system has a unique solution.

Solutions: The system has unique solution if and only if $a^2 - 4 \neq 0$. That is $a \neq \pm 2$.

5. (a) Compute

$$\begin{vmatrix} -2 & 0 & 0 & 0 \\ 5 & 3 & 5 & 7 \\ 3 & 0 & 2 & 1 \\ 8 & 0 & 2 & 2 \end{vmatrix}.$$

Solution: By cofactor's formula, we get

$$\det(A) = -2 \begin{vmatrix} 3 & 5 & 7 \\ 0 & 2 & 1 \\ 0 & 2 & 2 \end{vmatrix} = (-2)3 \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = -2 \cdot 3 \cdot 2 = -12.$$

(b) Compute A^{-1}, where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 5 \end{bmatrix}$$

Solution:

$$A^{-1} = \begin{bmatrix} -3 & -2 & 2 \\ -4 & -1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

6. Solve the following linear system using Cramer's rule.

$$\begin{aligned} -2x_1 + 3x_2 - x_3 &= 1 \\ x_1 + 2x_2 - x_3 &= 4 \\ -2x_1 - x_2 + x_3 &= -3 \end{aligned}$$

Solutions: By Cramer's rule.

$$|A| = \begin{vmatrix} -2 & 3 & -1 \\ 1 & 2 & -1 \\ -2 & -1 & 1 \end{vmatrix} = -2$$

$$|A_1| = \begin{vmatrix} 1 & 3 & -1 \\ 4 & 2 & -1 \\ -3 & -1 & 1 \end{vmatrix} = -4$$

$$|A_2| = \begin{vmatrix} -2 & 1 & -1 \\ 1 & 4 & -1 \\ -2 & -3 & 1 \end{vmatrix} = -6$$

$$|A_3| = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 2 & 4 \\ -2 & -1 & -3 \end{vmatrix} = -8$$

So

$$x_1 = \frac{|A_1|}{|A|} = 2,\ x_2 = \frac{|A_2|}{|A|} = 3,\ x_3 = \frac{|A_3|}{|A|} = 4.$$

7. Let $A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$

(a) Find a basis of column space of A.

Solutions The reduced echelon form of A is

$$\begin{pmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since it has pivots in first and third column, we conclude that first and third column of A, that is, $\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ is a basis of $\mathrm{Col}(A)$

(b) Find a basis of the null space of A.

Solution: From reduced echelon form of A, we can solve $AX = \vec{0}$. We have $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + 2x_5$. Hence the general solution of $AX = \vec{0}$ is

$$X = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

So $\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ is a basis of null space.

(c) Verify dimension theorem $\dim \mathrm{Nul} A + \mathrm{rank} A = n$ for this matrix A.

Solutions: From (1) and (2), we the rank of A is 2 and $\dim \mathrm{Nul}(A)=$ 3. So we get

$$\dim \mathrm{Nul} A + \mathrm{rank} A = 3 + 2 = 5 = n.$$