Math 265, Midterm 1

February 18th, 2019

Name: ______________________________

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	16	
2	16	
3	10	
4	15	
5	15	
6	13	
7	15	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either **T** or **F** in the table below. A, B, C, X, b are always matrices here (2 points each).

(a) AA^T is a square matrix.

(b) If $AB = AC$ and A is invertible then $B = C$.

(c) If a system of linear equations has 2 solutions then it has infinitely many solutions.

(d) Let A be a 3×5-matrix then the dimension of the null space $\mathrm{Nul}(A)$ could be 0.

(e) If $A^3 = 0$ then A^{-1} does not exist.

(f) Let A be an $n \times n$ matrix. If columns of A are linear dependent then $\det(A) = 0$

(g) There is only one basis $\{v_1, \ldots, v_n\}$ in $\mathbb{R}^n$, that is, the standard basis $\{e_1, \ldots, e_n\}$.

(h) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Then T is onto if and only if T is one-to-one.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Answer	T	T	T	F	T	T	F	T

2. Quick Questions (no details of explanation needed), A, B, C, X, b are always matrices here (4 points each):

(a) Suppose A is a 4×4-matrix and $\det(A) = 3$ then $\det(2A) =$?

Solutions: $\det(2A) = 2^4 \det(A) = 2^4 \cdot 3 = 48$

(b) Suppose that $A^2 = A$. $\det(A) =$?

Solutions: We have $\det(A)^2 = \det(A)$. So $\det(A) = 0$ or $\det(A) = 1$.

(c) Let $A = \begin{bmatrix} 1 & 1 \\ 0 & -2 \\ 1 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & -1 \end{bmatrix}$. Is $A^T B - 2I_2$ invertible? If so, find $(A^T B - 2I_2)^{-1}$.

Solutions: $A^T B - 2I_2 = \begin{pmatrix} 2 & 0 \\ -4 & 0 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -4 & -2 \end{pmatrix}$. The determinant $A^T B - 2I_2$ is 0. So it is not invertible.

(d) Let $\mathbb{Z}$ be the set of integers and $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x,\ y \in \mathbb{Z} \right\}$ be a subset of $\mathbb{R}^2$. Is V a subspace of $\mathbb{R}^2$? Why?

Solutions: No, V is not a subspace of $\mathbb{R}^2$ because it is not closed under scalar multiplication. For example, Let $r = \frac{1}{2}$, and $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in V$. But

$$\frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

is not in V.

3. Find an equation relating $a,\ b,\ c$ so that the following linear system is consistent for any values of $a,\ b,\ c$ that satisfy that equation (10 points):

$$\begin{aligned} x + 2y - 3z &= a \\ 2x + 3y + 3z &= b \\ 5x + 9y - 6z &= c \end{aligned}$$

Solutions:

The augmented matrix of the above system is

$$\begin{pmatrix} 1 & 2 & -3 & a \\ 2 & 3 & 3 & b \\ 5 & 9 & -6 & c \end{pmatrix}.$$

Then we get the echelon form:

$$\begin{pmatrix} 1 & 2 & -3 & a \\ 0 & -1 & 9 & b - 2a \\ 0 & 0 & 0 & c - b - 3a \end{pmatrix}.$$

Hence the system has solution if and only if $c - b - 3a = 0$.

4. (a) Consider the matrix

$$A = \begin{bmatrix} -2 & 7 & 6 & 8 \\ 0 & 0 & 3 & 0 \\ 0 & a & 2 & 1 \\ 0 & 9 & 2 & a \end{bmatrix}.$$

Find a so that A is invertible (8 points).

Solutions: By cofactor's formula on the first column, we get

$$\det(A) = -2 \begin{vmatrix} 0 & 3 & 0 \\ a & 2 & 1 \\ 9 & 2 & a \end{vmatrix}.$$

Use the cofactor formula again to the first row. We get

$$\det(A) = (-2)(-3) \begin{vmatrix} a & 1 \\ 9 & a \end{vmatrix} = 6(a^2 - 9).$$

Since A is invertible if and only if $\det(A) \neq 0$, we conclude that

$$6(a^2 - 9) \neq 0$$

which is equivalent to $a \neq \pm 3$.

(b) Compute A^{-1}, where

$$A = \begin{bmatrix} 2 & 2 & 5 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

(7 points)

Solution:

$$A^{-1} = \begin{bmatrix} 2 & -3 & -4 \\ 1 & -2 & -1 \\ -1 & 2 & 2 \end{bmatrix}$$

5. Solve the following linear system using Cramer's rule (15 points).

$$\begin{aligned} 2x_1 \quad &+3x_2 - x_3 &&= -5 \\ x_1 \quad &+2x_3 &&= 0 \\ x_1 \quad &+2x_2 + 3x_3 &&= 1 \end{aligned}$$

Solutions: By Cramer's rule.

$$|A| = \begin{vmatrix} 2 & 3 & -1 \\ 1 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -13$$

$$|A_1| = \begin{vmatrix} -5 & 3 & -1 \\ 0 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix} = 26$$

$$|A_2| = \begin{vmatrix} 2 & -5 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 3 \end{vmatrix} = 0$$

$$|A_3| = \begin{vmatrix} 2 & 3 & -5 \\ 1 & 0 & 0 \\ 1 & 2 & 1 \end{vmatrix} = -13$$

So

$$x_1 = \frac{|A_1|}{|A|} = -2, \; x_2 = \frac{|A_2|}{|A|} = 0, \; x_3 = \frac{|A_3|}{|A|} = 1.$$

6. Let $A = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 1 & 2 & 0 & -1 \\ 2 & 4 & -1 & 0 \end{pmatrix}$.

(a) Find a basis for the column space of A (5 points).

Solutions: We compute the reduced echelon form of A is

$$\begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The 1st 3rd columns of the reduced echelon form have the leading ones. So the first two columns $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ of A forms a basis of the column space of A.

(b) Find a basis for the null space of A (5 points).

Solutions: From the reduced echelon form, we get equations $x_1 + 2x_2 - x_4 = 0$ and $x_3 - 2x_4 = 0$. Therefore, we have $x_1 = -2x_2 + x_4$ and $x_3 = 2x_4$. Hence, any vector in $N(A)$ can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2x_2 + x_4 \\ x_2 \\ 2x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

with x_2, x_4 free parameters. So $\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}$ forms a basis of the null space.

(c) Verify the equality $\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = n$ (3 points).

Solutions: We have $\operatorname{rank}(A) = 2$ and $\dim \operatorname{Nul}(A) = 2$ from the above. So

$$\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = 2 + 2 = 4 = n.$$

7. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation. Suppose that

$$T(\begin{bmatrix}1\\1\end{bmatrix}) = \begin{bmatrix}1\\0\\3\end{bmatrix} \text{ and } T(\begin{bmatrix}-1\\1\end{bmatrix}) = \begin{bmatrix}-1\\2\\1\end{bmatrix}.$$

(a) Find the standard matrix of T (5 points).

Solutions: We easily find that $\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}$. So

$$T(\begin{bmatrix}1\\0\end{bmatrix}) = \frac{1}{2}T(\begin{bmatrix}1\\1\end{bmatrix}) + \frac{1}{2}T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}1\\-1\\1\end{bmatrix}.$$

and

$$T(\begin{bmatrix}0\\1\end{bmatrix}) = \frac{1}{2}T(\begin{bmatrix}1\\1\end{bmatrix}) - \frac{1}{2}T(\begin{bmatrix}1\\-1\end{bmatrix}) = \begin{bmatrix}0\\1\\2\end{bmatrix}.$$

Therefore the standard matrix of T is given by

$$(T(e_1), T(e_2)) = \begin{bmatrix}1 & 0\\-1 & 1\\1 & 2\end{bmatrix}$$

(b) Is T onto? explain (5 points).

Solutions: The echelon form of the standard matrix is $\begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$. Since there is a zero row in the echelon form, T is not onto.

(c) Is T one-to-one? explain (5 points).

Solutions: Since the echelon form has pivots on each column, T is indeed one-to-one.