Math 265, Practice Midterm 2

Name: ____

This exam consists of 7 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	16	
2	16	
3	16	
4	20	
5	16	
6	16	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) If rows of A are linearly dependent so are columns.

(b) Let V be a vector space with basis $\mathcal{B}$ and $\dim V = n$. Then $v_1, \ldots, v_m \in V$ is a basis of V if and only if $[v_1]_{\mathcal{B}}, \ldots, [v_m]_{\mathcal{B}}$ is a basis of $\mathbb{R}^n$.

(c) If A has an eigenvalue 0 then A is not invertible.

(d) A is diagonalizable if and only if all eigenvalues of A are distinct.

(e) Let A be a 3×5-matrix then Nul(A) has positive dimension.

(f) Let k_i be the multiplicity of an eigenvalue λ_i of A. If $k_i \geq 2$ then A is not diagonalizable.

(g) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, $\mathcal{B}$ basis of $\mathbb{R}^n$ and A the standard matrix of T. Then A is similar to $[T]_{\mathcal{B}}$.

(h) Let $A \in \mathbb{R}^{n \times n}$ be a real matrix. If $a + bi$ is an eigenvalue of A so is $a - bi$.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Answer	F	T	T	F	T	F	T	T

2. Quick Questions, $A,\ B,\ C,\ X,\ b$ are always matrices here:

(a) Suppose that $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ with eigenvalues $3, 1$. Compute A^k.

Solutions: A is diagonalizable $A = P \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} P^{-1}$ with $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Then $A^k = P \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}$

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ with the standard matrix $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Let S be a parallelogram with area 2. Compute the area of $T(S)$.

Solutions: The determinate of the standard matrix is 5. So Area $T(S) = 5$ Area of $S = 10$.

(c) Find a basis of row space of $A = \begin{pmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{pmatrix}$

Solutions: An echelon form of A is $\begin{pmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ So, $\begin{pmatrix} 1 & -2 & 2 & 3 & -1 \end{pmatrix}$, $(0\ 0\ 5\ 10\ -10)$ forms a basis of row space of A.

3. Let consider the following subset $V \subset \mathbb{R}^{2\times 2}$.

$$V = \left\{ \begin{bmatrix} x & y \\ z & w \end{bmatrix} | x + 2z = 0, \ -y + w = 0 \right\}$$

(a) Show that V is a subspace of $\mathbb{R}^{2\times 2}$.

Proof: Given $A_1 = \begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix} \in V$, $A_2 := \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix} \in V$ and $c \in \mathbb{R}$. We have

$$A_1 + A_2 = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ z_1 + z_2 & w_1 + w_2 \end{bmatrix}, \quad cA_1 = \begin{bmatrix} cx_1 & cy_1 \\ cz_1 & cw_1 \end{bmatrix}.$$

It is clear that we have $(x_1 + x_2) + 2(z_1 + z_2) = 0$ and $-(y_1 + y_2) + (w_1 + w_2) = 0$. So $A_1 + A_2$ is in V. Also $(cx_1) + 2(cz_1) = 0$ and $-(cy_1) + (cw_1) = 0$. So $cA_1 \in V$. Hence V is close under addition and scalar multiplication. Hence V is a subspace of $\mathbb{R}^{2\times 2}$.

(b) Find a basis of V.

Solutions: It we identify $\mathbb{R}^{2\times 2}$ with $\mathbb{R}^4$ via $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$. Then V is just the space

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} | x + 2z = 0, \ -y + w = 0 \right\}$$

Namely, V is just the null space 0f $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$, which we can find a basis of this null space:

$$\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

So $\begin{bmatrix} -2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ is a basis of V.

(c) Find the dimension of V.

Solutions: As explained as the above, $\dim V = 2$.

4. Let

$$A = \begin{pmatrix} 2 & a & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

(a) Determine all values of a such that A is diagonalizable.

Solutions: The characteristic polynomial of A is $f_A(t) = \begin{vmatrix} 2-t & a & 1 \\ 0 & 2-t & -1 \\ 0 & 0 & 3-t \end{vmatrix} = -(t-2)^2(t-3)$. So we get eigenvalues $\lambda_1 = 2$ with multiplicity $k_1 = 2$, $\lambda_2 = 3$ with multiplicity $k_2 = 1$.

For $\lambda_1 = 2$, we solve $(A - 2I_3)X = 0$ to find a basis of eigenspace E_2. Note that $A - 2I_3 = \begin{pmatrix} 0 & a & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}$. We see that $\text{rank}(A - 2I_3) = 2$ when $a \neq 0$ and is 1 when $a = 0$. So when $a \neq 0$, $\dim E_2 = 3 - 2 = 1 < k_1 = 2$ which implies that A is not diagonalizable.

Now if $a = 0$ then $\dim E_2 = 2 = k_2$. For $\lambda_2 = 3$, we have $A - 3I_3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. We easily calculate that E_3 is dimensional 1 spanned by $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. So $\dim E_3 = 1 = k_2$. Thus only when $a = 0$ then A is diagonalizable.

(b) When a is diagonalizable. Diagonalize A.

Solutions: When $a = 0$, we find basis of E_2 by solving $(A - 2I_3)X = 0$:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, $A = P\Lambda P^{-1}$ with

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

5. Let $\mathbb{P}_2[t] = \{f(t) = a_2t^2 + a_1t + a_0\}$ be the space of polynomials with maximal degree 2. Define the following map $T : \mathbb{P}_2[t] \to \mathbb{R}^2$ via $T(f) = \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix}$.

(a) Show T is a linear transformation.

Proof: Given $f(t),\ g(t) \in BP_2[t]$, and $c \in \mathbb{R}$,

$$T(f+g) = \begin{bmatrix} (f+g)(-1) \\ (f+g)(1) \end{bmatrix} = \begin{bmatrix} f(-1) + g(-1) \\ f(1) + g(1) \end{bmatrix} = \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix} + \begin{bmatrix} g(-1) \\ g(1) \end{bmatrix} = T(f) + T(g).$$

Similarly, we see that $T(cf) = \begin{bmatrix} (cf)(-1) \\ (cf)(1) \end{bmatrix} = c \begin{bmatrix} f(-1) \\ f(1) \end{bmatrix} = cT(f)$. So T is a linear transformation.

(b) Let $\mathcal{B}$ be a standard basis of $\mathbb{P}_2[t]$ and $\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. Find matrix M of T relative to basis $\mathcal{B}$ and $\mathcal{C}$.

Solutions: By definition,

$$M = \left[[T(1)]_{\mathcal{C}}, [T(t)]_{\mathcal{C}}, [T(t^2)]_{\mathcal{C}} \right] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{C}}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}_{\mathcal{C}}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{C}} \right] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(c) Let $f = t^2 + t + 1$. Find $[T(f)]_{\mathcal{C}}$.
Solutions: We know that $[T(f)]_{\mathcal{C}} = M[f]_{\mathcal{B}}$. So

$$[T(f)]_{\mathcal{C}} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

6. Let $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots a_1 t + a_0$ be polynomial and $A \in \mathbb{R}^{m\times m}$ be an $m \times m$-matrix. Then we define

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_m.$$

In particular, $f(A)$ is still an $m \times m$-matrix.

(a) If A is similar to B, that is $A = PBP^{-1}$, show that $f(A) = Pf(B)P^{-1}$

Proof Since $A = PBP^{-1}$, we have $A^k = PB^kP^{-1}$. Therefore

$$\begin{aligned} f(A) &= a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_m \\ &= a_n PB^nP^{-1} + a_{n-1} PB^{n-1}P^{-1} + \cdots + a_1 PBP^{-1} + a_0 PI_nP^{-1} \\ &= P(a_n B^n + a_{n-1} B^{n-1} + \cdots + a_1 B + a_0 I_n)P^{-1} \\ &= Pf(B)P^{-1}. \end{aligned}$$

(b) If A is diagonal matrix and $f(t) = f_A(t)$ is characteristic polynomial of A, show that $f(A)$ is a zero matrix.

Proof: Note the multiplication and addition of *diagonal* matrices are just multiplications and additions entry by entry. So if $A = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$ then $f(A) = \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix}$ In this case, $f(t) = f_A(t) = (t - \lambda_1) \cdots (t - \lambda_n)$. So $f(\lambda_1) = \cdots f(\lambda_n) = 0$. Hence $f(A) = 0$.

(c) If A diagonalizable matrix and $f(t) = f_A(t)$ is characteristic polynomial of A, show that $f(A)$ is a zero matrix.

Proof:

Since A is diagonalizable, $A = P\Lambda P^{-1}$ with $\Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$. Then by (1), we have

$$f(A) = Pf(\Lambda)P^{-1} = P \begin{bmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{bmatrix} P^{-1}.$$

Since $f(t) = f_A(t) = f_\Lambda(t) = (\lambda_1 - t) \cdots (\lambda_n - t)$, $f(A) = P0P^{-1} = 0$.