Math 265, Midterm 2

Name: ______________________________

This exam consists of 7 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	16	
2	16	
3	16	
4	18	
5	16	
6	18	
Total	100	

Notations: $\mathbb{R}$ denotes the set of real number; $\mathbb{C}$ denotes the set of complex numbers. $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$-matrices with entries in $\mathbb{R}$; $\mathbb{R}^n = \mathbb{R}^{n\times 1}$ denotes the set of n-column vectors; Similar notations for matrices of complex numbers $\mathbb{C}$; $\mathbb{P}_n[t]$ denotes the set of polynomials with coefficients in $\mathbb{R}$ and the most degree n, that is,

$$\mathbb{P}_n[t] = \{f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0, \quad a_i \in \mathbb{R}, \ \forall i\}.$$

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here. (2 points each)

(a) If a matrix A has r linearly dependent rows then it has also r linearly independent columns.

(b) Let A be 4×5-matrix with rank 4 then the linear system $AX = b$ always has infinity many solutions for any $b \in \mathbb{R}^4$.

(c) If A is similar to B then A and B share the same eigenvectors.

(d) Let v_1 and v_2 be eigenvectors with eigenvalues 0 and 1 respectively then v_1 and v_2 must be linearly independent.

(e) Let $A \in \mathbb{C}^{n\times n}$ be a complex matrix. Then $v \in \mathbb{C}^n$ is an eigenvector of A if and only if the complex conjugation $\bar{v}$ is.

(f) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Then T sends a parallelogram in $\mathbb{R}^2$ to a parallelogram in $\mathbb{R}^2$ with the same area.

(g) Let k_i be the multiplicity of an eigenvalue λ_i of A. If A is not diagonalizable then one of $k_i \geq 2$.

(h) Let V be a vector space with basis $\mathcal{B}$ and $\dim V = n$. Then $w_1, \ldots, w_n \in V$ is a basis of V if and only if $[w_1]_{\mathcal{B}}, \ldots, [w_n]_{\mathcal{B}}$ spans $\mathbb{R}^n$.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Answer	F	T	F	T	F	F	T	T

2. Quick Questions, $A,\ B,\ C,\ X,\ b$ are always matrices here:

(a) Suppose that $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ with one eigenvalue $1+i$. Compute A^k. (6 points)

Solutions: We easily see that A has an eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ with the eigenvalue $1+i$. Since A is a real matrix, A also has an eigenvector $\begin{bmatrix} 1 \\ i \end{bmatrix}$ with the eigenvalue $1-i$. Thus A is is diagonalizable $A = P \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix} P^{-1}$ with $P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$. Then

$$A^k = P \begin{bmatrix} (1+i)^k & 0 \\ 0 & (1-i)^k \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} (1+i)^k & 0 \\ 0 & (1-i)^k \end{bmatrix} \frac{1}{2i} \begin{bmatrix} i & -1 \\ i & 1 \end{bmatrix}.$$

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with the standard matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Given another basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$. Find matrix $[T]_{\mathcal{B}}$ of T relative to basis $\mathcal{B}$. (5 points)

Solutions: $[T]_{\mathcal{B}} = P^{-1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} P$ with $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Since $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. We have $[T]_{\mathcal{B}} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

(c) Find a basis of the following subspace of $\mathbb{R}^{2\times 2}$: Here A^T means transpose of A. (5 points)

$$W = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} | A^T = A \right\}.$$

Solutions: $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Therefore $A^T = A$ is equivalent to $b = c$. So

$$W = \left\{ A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \ a, b, d \text{ are arbitrary} \right\}.$$

Since

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

we see that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ forms a basis of W.

3. Let $\mathbb{R}_5 := \{(a_1\ a_2\ a_3\ a_4\ a_5) | a_i \in \mathbb{R}\}$ be the space of row vectors. Consider the following set of row vectors

$$v_1 = (1\ 0\ -2\ 1\ 0),\ v_2 = (-1\ 1\ 0\ 2\ 1),\ v_3 = (1\ 1\ -4\ 4\ 1)$$

(a) Find a basis of $\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\} \subset \mathbb{R}_5$. (6 points)

Solutions: Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 1 & 1 & -4 & 4 & 1 \end{bmatrix}$. To find a basis of $\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$, it suffices to find a basis of row space of A. An echelon form of A is $\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. As nonzero rows of echelon form forms a basis of Row(A), we get a basis of $\mathrm{Span}\{\mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3\}$:

$$(1\ 0\ -2\ 1\ 0),\quad (0\ 1\ -2\ 3\ 1)$$

(b) Is $\{v_1, v_2, v_3\}$ linearly independent? Explain. (4 points)

Solutions: If v_1, v_2, v_3 is linearly independent then v_1, v_2, v_3 forms another basis of Row(A). This means that rank(A) would be 3. But the above calculation show that A has rank 2. So v_1, v_2, v_3 is linearly dependent.

(c) Is $w = (1\ 1\ 1\ 1\ 1)$ in $\mathrm{Span}\{v_1, v_2, v_3\}$? Explain. (6 points)

Solutions: Consider matrix $A' = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 1 & 1 & -4 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ which add row w to A. Note that w is in $\mathrm{Span}\{v_1, v_2, v_3\}$ if and only if $\mathrm{Row}(\mathrm{A}') = \mathrm{Row}(\mathrm{A})$, and this is equivalent to that $\mathrm{rank}(A) = \mathrm{rank}(A')$. But we can easily calculate that A' has rank 3 (by computing echelon form of A'), not $2 = \mathrm{rank}(\mathrm{A})$. So w is not in $\mathrm{Span}\{v_1, v_2, v_3\}$.

4. Let

$$A = \begin{bmatrix} 2 & 0 & 0 \\ b & 2 & a \\ 3 & 0 & 3 \end{bmatrix}$$

(a) Find equations of a and b so that if a, b satisfies the equation then A is diagonalizable. (10 points)

Solutions: The characteristic polynomial of A is $f_A(t) = \begin{vmatrix} 2-t & 0 & 0 \\ b & 2-t & a \\ 3 & 0 & 3-t \end{vmatrix} = -(t-2)^2(t-3)$. So we get eigenvalues $\lambda_1 = 2$ with multiplicity $k_1 = 2$, $\lambda_2 = 3$ with multiplicity $k_2 = 1$.

For $\lambda_1 = 2$, we solve $(A - 2I_3)X = 0$ to find a basis of eigenspace E_2. Note that $A - 2I_3 = \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & a \\ 3 & 0 & 1 \end{pmatrix}$, which has echelon form $\begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 0 & a - \frac{b}{3} \\ 0 & 0 & 0 \end{pmatrix}$.

Now we have two situations of rank$(A - 2I_3)$: Case 1: $a - \frac{b}{3} \neq 0$, then rank$(A - 2I_3) = 2$ and consequently $\dim E_2 = 3 - 2 = 1 < k_1 = 2$. And hence A is not diagonalizable.

Case 2: $a - \frac{b}{3} = 0$, then rank$(A - 2I_3) = 1$ and consequently $\dim E_2 = 3 - 1 = 2 = k_1 = 2$.

For $\lambda_2 = 3$, we have $A - 3I_3 = \begin{pmatrix} -1 & 0 & 0 \\ b & -1 & a \\ 3 & 0 & 0 \end{pmatrix}$. We easily calculate that E_3 is dimensional 1 spanned by $\begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix}$. So $\dim E_3 = 1 = k_2$.

In summary, only when $b = 3a$ then A is diagonalizable.

(b) When A is diagonalizable. Diagonalize A. (8 points)

Solutions: When $b = 3a$, we find basis of E_2 by solving $(A - 2I_3)X = 0$:

$$\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Therefore, $A = P\Lambda P^{-1}$ with

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ -3 & 0 & 1 \end{pmatrix} \text{ and } \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

5. Let $\mathbb{P}_2[t] = \{f(t) = a_2t^2 + a_1t + a_0\}$ be the space of polynomials with maximal degree 2. Define the following map $T : \mathbb{P}_2[t] \to \mathbb{P}_2[t]$ via

$$T(f(t)) = f'(t) + 2f(t).$$

(a) Show T is a linear transformation. (5 points)

Proof: Given $f(t),\ g(t) \in BP_2[t]$, and $c \in \mathbb{R}$,

$$T(f+g) = (f+g)' + 2(f+g) = f' + 2f + g' + 2g = T(f) + T(g).$$

Similarly, we see that $T(cf) = (cf)' + 2(cf) = c(f' + 2f) = cT(f)$. So T is a linear transformation.

(b) Let $\mathcal{B}$ be a standard basis of $\mathbb{P}_2[t]$ and find matrix M of T relative to basis $\mathcal{B}$. (6 points)

Solutions: Note that $\mathcal{B} = \{1,\ t,\ t^2\}$ By definition,

$$M = \left[[T(1)]_{\mathcal{B}}, [T(t)]_{\mathcal{C}}, [T(t^2)]_{\mathcal{B}}\right] = \left[[2]_{\mathcal{B}},\ [1+2t]_{\mathcal{B}},\ [2t+2t^2]_{\mathcal{B}}\right] = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

(c) Find an $f \in \mathbb{P}_2[t]$ and $\lambda \in \mathbb{R}$ so that $[T(f)]_{\mathcal{B}} = \lambda[f]_{\mathcal{B}}$. (5 points)
Solutions: We know that $[T(f)]_{\mathcal{B}} = M[f]_{\mathcal{B}}$. Let $v = [f]_{\mathcal{B}} \in \mathbb{R}^3$. Then $[T(f)]_{\mathcal{B}} = \lambda[f]_{\mathcal{B}}$ is equivalent to $Mv = \lambda v$. Namely, v is an eigenvector of M with eigenvalue λ. As M is an upper triangular matrix, we see eigenvalues of M is just 2. So $\lambda = 2$. Solve $(M - 2I_3)X = \vec{0}$, we find that $v = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So $f = k(1 + 0 \cdot t + 0 \cdot t^2) = k$.

6. A matrix $A \in \mathbb{R}^{n\times n}$ is called *nilpotent* if $A^m = 0$ for some $m > 0$.

(a) Let $v \in \mathbb{R}^n$ be an eigenvector of A with eigenvalue λ. Show that v is an eigenvector of A^m with eigenvalue λ^m. (5 points)

Proof: Since $Av = \lambda v$, we have

$$A^m v = A^{m-1}(Av) = A^{m-1}(\lambda v) = \lambda A^{m-1} v = \lambda A^{m-2}(Av) = \cdots = \lambda^{m-1} Av = \lambda^m v.$$

As $v \neq \vec{0}$, v is an eigenvector of A^m with eigenvalue λ^m.

(b) Show that if A is nilpotent then all eigenvalues of A are 0. (5 points)

Proof: Let λ be an eigenvalue of A. Then (a) shows that λ^m is an eigenvalue of A^m. But $A^m = 0$ which only has eigenvalue 0. Thus $\lambda^m = 0$. So $\lambda = 0$.

(c) Give an example of nilpotent matrix $A \neq 0$. (3 points)

Solutions: $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that $A^2 = 0$.

(d) Show that a nilpotent matrix is *not* diagonalizable unless $A = 0$. (5 points)

Proof: Suppose that A is diagonalizable and nilpotent. Then $A = P\Lambda P^{-1}$ where Λ is a diagonal matrix with eigenvalues of A on the diagonal. But (b) shows that all eigenvalues of A are zeros. Hence $\Lambda = 0$. So $A = P\Lambda P^{-1} = P0P^{-1} = 0$. Therefore nilpotent matrix A is not diagonalizable unless $A = 0$.