Math 265, Midterm 2

Name: ________________________________

This exam consists of 7 pages including this front page.

Ground Rules

1. No calculator is allowed.

2. Show your work for every problem unless otherwise stated.

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Notations: \( \mathbb{R} \) denotes the set of real number; \( \mathbb{C} \) denotes the set of complex numbers. \( \mathbb{R}^{m \times n} \) denotes the set of \( m \times n \)-matrices with entries in \( \mathbb{R} \); \( \mathbb{R}^n = \mathbb{R}^{n \times 1} \) denotes the set of \( n \)-column vectors; Similar notations for matrices of complex numbers \( \mathbb{C} \); \( \mathbb{P}_n[t] \) denotes the set of polynomials with coefficients in \( \mathbb{R} \) and the most degree \( n \), that is,

\[
\mathbb{P}_n[t] = \{ f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0, \ a_i \in \mathbb{R}, \ \forall i \}.
\]

1. The following are true/false questions. You don’t have to justify your answers. Just write down either T or F in the table below. \( A, B, C, X, b \) are always matrices here. (2 points each)

(a) If a matrix \( A \) has \( r \) linearly dependent rows then it has also \( r \) linearly independent columns.
(b) Let \( A \) be 4 \times 5-matrix with rank 4 then the linear system \( AX = b \) always has infinity many solutions for any \( b \in \mathbb{R}^4 \).
(c) If \( A \) is similar to \( B \) then \( A \) and \( B \) share the same eigenvectors.
(d) Let \( v_1 \) and \( v_2 \) be eigenvectors with eigenvalues 0 and 1 respectively then \( v_1 \) and \( v_2 \) must be linearly independent.
(e) Let \( A \in \mathbb{C}^{n \times n} \) be a complex matrix. Then \( v \in \mathbb{C}^n \) is an eigenvector of \( A \) if and only if the complex conjugation \( \bar{v} \) is.
(f) Let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be a linear transformation. Then \( T \) sends a parallelogram in \( \mathbb{R}^2 \) to a parallelogram in \( \mathbb{R}^2 \) with the same area.
(g) Let \( k_i \) be the multiplicity of an eigenvalue \( \lambda_i \) of \( A \). If \( A \) is not diagonalizable then one of \( k_i \geq 2 \).
(h) Let \( V \) be a vector space with basis \( \mathcal{B} \) and dim \( V = n \). Then \( w_1, \ldots, w_n \in V \) is a basis of \( V \) if and only if \([w_1]_{\mathcal{B}}, \ldots, [w_n]_{\mathcal{B}}\) spans \( \mathbb{R}^n \).

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2. Quick Questions, $A$, $B$, $C$, $X$, $b$ are always matrices here:

(a) Suppose that $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ with one eigenvalue $1 + i$. Compute $A^k$. (6 points)

Solutions: We easily see that $A$ has an eigenvector $\begin{bmatrix} 1 \\ -i \end{bmatrix}$ with the eigenvalue $1 + i$. Since $A$ is a real matrix, $A$ also has an eigenvector $\begin{bmatrix} 1 \\ i \end{bmatrix}$ with the eigenvalue $1 - i$. Thus $A$ is diagonalizable $A = P \begin{bmatrix} 1 + i & 0 \\ 0 & 1 - i \end{bmatrix} P^{-1}$ with $P = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}$. Then

$$A^k = P \begin{bmatrix} (1 + i)^k & 0 \\ 0 & (1 - i)^k \end{bmatrix} P^{-1} = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} (1 + i)^k & 0 \\ 0 & (1 - i)^k \end{bmatrix} \frac{1}{2i} \begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}.$$

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with the standard matrix $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Given another basis $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ of $\mathbb{R}^2$. Find matrix $[T]_B$ of $T$ relative to basis $B$. (5 points)

Solutions: $[T]_B = P^{-1} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} P$ with $P = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Since $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. We have $[T]_B = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$.

(c) Find a basis of the following subspace of $\mathbb{R}^{2 \times 2}$: Here $A^T$ means transpose of $A$. (5 points)

$$W = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid A^T = A \right\}.$$

Solutions: $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Therefore $A^T = A$ is equivalent to $b = c$. So

$$W = \left\{ A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \mid a, b, d \text{ are arbitrary} \right\}.$$

Since

$$\begin{bmatrix} a & b \\ b & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$

we see that $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ forms a basis of $W$. 

3
3. Let $\mathbb{R}_5 := \{(a_1 a_2 a_3 a_4 a_5) | a_i \in \mathbb{R}\}$ be the space of row vectors. Consider the following set of row vectors $v_1 = (1 \ 0 \ -2 \ 1 \ 0), \ v_2 = (-1 \ 1 \ 0 \ 2 \ 1), \ v_3 = (1 \ 1 \ -4 \ 4 \ 1)$

(a) Find a basis of $\text{Span}\{v_1, v_2, v_3\} \subset \mathbb{R}_5$. (6 points)

*Solutions:* Consider the matrix $A = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 1 & 1 & -4 & 4 & 1 \end{bmatrix}$. To find a basis of $\text{Span}\{v_1, v_2, v_3\}$, it suffices to find a basis of row space of $A$.

An echelon form of $A$ is $\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -2 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. As nonzero rows of echelon form forms a basis of $\text{Row}(A)$, we get a basis of $\text{Span}\{v_1, v_2, v_3\}$:

$$(1 \ 0 \ -2 \ 1 \ 0), \ (0 \ 1 \ -2 \ 3 \ 1)$$

(b) Is $\{v_1, v_2, v_3\}$ linearly independent? Explain. (4 points)

*Solutions:* If $v_1, v_2, v_3$ is linearly independent then $v_1, v_2, v_3$ forms another basis of $\text{Row}(A)$. This means that $\text{rank}(A)$ would be 3. But the above calculation show that $A$ has rank 2. So $v_1, v_2, v_3$ is linearly dependent.

(c) Is $w = (1 \ 1 \ 1 \ 1 \ 1)$ in $\text{Span}\{v_1, v_2, v_3\}$? Explain. (6 points)

*Solutions:* Consider matrix $A' = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 1 & 1 & -4 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$ which add row $w$ to $A$. Note that $w$ is in $\text{Span}\{v_1, v_2, v_3\}$ if and only if $\text{Row}(A') = \text{Row}(A)$, and this is equivalent to that $\text{rank}(A) = \text{rank}(A')$. But we can easily calculate that $A'$ has rank 3 (by computing echelon form of $A'$), not 2 = $\text{rank}(A)$. So $w$ is not in $\text{Span}\{v_1, v_2, v_3\}$. 

4
4. Let

\[ A = \begin{bmatrix} 2 & 0 & 0 \\ b & 2 & a \\ 3 & 0 & 3 \end{bmatrix} \]

(a) Find equations of \( a \) and \( b \) so that if \( a, b \) satisfies the equation then \( A \) is diagonalizable. (10 points)

\textbf{Solutions:} The characteristic polynomial of \( A \) is

\[ f_A(t) = \begin{vmatrix} 2 - t & 0 & 0 \\ b & 2 - t & a \\ 3 & 0 & 3 - t \end{vmatrix} = -(t - 2)^2(t - 3). \]

So we get eigenvalues \( \lambda_1 = 2 \) with multiplicity \( k_1 = 2 \), \( \lambda_2 = 3 \) with multiplicity \( k_2 = 1 \).

For \( \lambda_1 = 2 \), we solve \((A - 2I_3)X = 0\) to find a basis of eigenspace \( E_2 \).

Note that \( A - 2I_3 = \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & a \\ 3 & 0 & 1 \end{pmatrix} \), which has echelon form \( \begin{pmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 0 & a - \frac{b}{3} \\ 0 & 0 & 0 \end{pmatrix} \).

Now we have two situations of \( \text{rank}(A - 2I_3) \): Case 1: \( a - \frac{b}{3} \neq 0 \), then \( \text{rank}(A - 2I_3) = 2 \) and consequently \( \dim E_2 = 3 - 2 = 1 < k_1 = 2 \). And hence \( A \) is not diagonalizable.

Case 2: \( a - \frac{b}{3} = 0 \), then \( \text{rank}(A - 2I_3) = 1 \) and consequently \( \dim E_2 = 3 - 1 = 2 = k_1 = 2 \).

For \( \lambda_2 = 3 \), we have \( A - 3I_3 = \begin{pmatrix} -1 & 0 & 0 \\ b & -1 & a \\ 3 & 0 & 0 \end{pmatrix} \). We easily calculate that \( E_3 \) is dimensional 1 spanned by \( \begin{pmatrix} a \\ 1 \end{pmatrix} \). So \( \dim E_3 = 1 = k_2 \).

In summary, only when \( b = 3a \) then \( A \) is diagonalizable.

(b) When \( A \) is diagonalizable. Diagonalize \( A \). (8 points)

\textbf{Solutions:} When \( b = 3a \), we find basis of \( E_2 \) by solving \((A - 2I_3)X = 0\):

\[ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \]

Therefore, \( A = P\Lambda P^{-1} \) with

\[ P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a \\ -3 & 0 & 1 \end{pmatrix} \] and \( \Lambda = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \).
5. Let $\mathbb{P}_2[t] = \{ f(t) = a_2t^2 + a_1t + a_0 \}$ be the space of polynomials with maximal degree 2. Define the following map $T : \mathbb{P}_2[t] \to \mathbb{P}_2[t]$ via
\[
T(f(t)) = f'(t) + 2f(t).
\]

(a) Show $T$ is a linear transformation. (5 points)

Proof: Given $f(t), g(t) \in BP_2[t]$, and $c \in \mathbb{R}$,

\[
T(f + g) = (f + g)' + 2(f + g) = f' + 2f + g' + 2g = T(f) + T(g).
\]

Similarly, we see that $T(cf) = (cf)' + 2(cf) = c(f' + 2f) = cT(f)$. So $T$ is a linear transformation.

(b) Let $\mathcal{B}$ be a standard basis of $\mathbb{P}_2[t]$ and find matrix $M$ of $T$ relative to basis $\mathcal{B}$. (6 points)

Solutions: Note that $\mathcal{B} = \{1, t, t^2\}$ By definition,

\[
M = \begin{bmatrix} [T(1)]_\mathcal{B}, [T(t)]_\mathcal{B}, [T(t^2)]_\mathcal{B} \end{bmatrix} = \begin{bmatrix} [2]_\mathcal{B}, [1 + 2t]_\mathcal{B}, [2t + 2t^2]_\mathcal{B} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}.
\]

(c) Find an $f \in \mathbb{P}_2[t]$ and $\lambda \in \mathbb{R}$ so that $[T(f)]_\mathcal{B} = \lambda [f]_\mathcal{B}$. (5 points)

Solutions: We know that $[T(f)]_\mathcal{B} = M[f]_\mathcal{B}$. Let $v = [f]_\mathcal{B} \in \mathbb{R}^3$. Then $[T(f)]_\mathcal{B} = \lambda [f]_\mathcal{B}$ is equivalent to $Mv = \lambda v$. Namely, $v$ is an eigenvector of $M$ with eigenvalue $\lambda$. As $M$ is an upper triangular matrix, we see eigenvalues of $M$ is just 2. So $\lambda = 2$. Solve $(M - 2I_3)X = \vec{0}$, we find that $v = k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So $f = k(1 + 0 \cdot t + 0 \cdot t^2) = k$. 


6. A matrix $A \in \mathbb{R}^{n \times n}$ is called \textit{nilpotent} if $A^m = 0$ for some $m > 0$.

(a) Let $v \in \mathbb{R}^n$ be an eigenvector of $A$ with eigenvalue $\lambda$. Show that $v$ is an eigenvector of $A^m$ with eigenvalue $\lambda^m$. (5 points)

\textit{Proof:} Since $Av = \lambda v$, we have

$$A^m v = A^{m-1}(Av) = A^{m-1}(\lambda v) = \lambda A^{m-1}v = \lambda A^{m-2}(Av) = \cdots = \lambda^m Av = \lambda^m v.$$

As $v \neq \vec{0}$, $v$ is an eigenvector of $A^m$ with eigenvalue $\lambda^m$.

(b) Show that if $A$ is nilpotent then all eigenvalues of $A$ are 0. (5 points)

\textit{Proof:} Let $\lambda$ be an eigenvalue of $A$. Then (a) shows that $\lambda^m$ is an eigenvalue of $A^m$. But $A^m = 0$ which only has eigenvalue 0. Thus $\lambda^m = 0$. So $\lambda = 0$.

(c) Give an example of nilpotent matrix $A \neq 0$. (3 points)

\textit{Solutions:} $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that $A^2 = 0$.

(d) Show that a nilpotent matrix is \textit{not} diagonalizable unless $A = 0$. (5 points)

\textit{Proof:} Suppose that $A$ is diagonalizable and nilpotent. Then $A = P\Lambda P^{-1}$ where $\Lambda$ is a diagonal matrix with eigenvalues of $A$ on the diagonal. But (b) shows that all eigenvalues of $A$ are zeros. Hence $\Lambda = 0$. So $A = P\Lambda P^{-1} = P0P^{-1} = 0$. Therefore nilpotent matrix $A$ is not diagonalizable unless $A = 0$. 

7