QUIZ 4

Solve the following initial value problem by using Laplace transform:

$$y'' + 2y' + 2y = 1, \;\; y(0) = 0, \;\; y'(0) = 1.$$

Solutions: Applying Laplace transform to the both sides of the equation. Set $Y(s) = \mathfrak{L}(y(t))$. Note that $\mathfrak{L}(y'') = s^2Y(s) - sy(0) - y'(0)$ and $\mathfrak{L}(y') = sY(s) - y(0)$. We obtain

$$s^2Y(s) - 1 + 2sY(s) + 2Y(s) = \mathfrak{L}(1) = \frac{1}{s}.$$

That is

$$Y(s) = \frac{1}{s^2 + 2s + 2} + \frac{1}{s(s^2 + 2s + 2)}.$$

To decompose $\frac{1}{s(s^2+2s+2)}$, we can write

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{a}{s} + \frac{bs + c}{(s^2 + 2s + 2)}$$

with a, b, c unknown coefficients. It is not hard to find that

$$\frac{1}{s(s^2 + 2s + 2)} = \frac{1}{2}\left(\frac{1}{s} - \frac{s+2}{s^2 + 2s + 2}\right) = \frac{1}{2}\left(\frac{1}{s} - \frac{s+1}{s^2 + 2s + 2} - \frac{1}{s^2 + 2s + 2}\right)$$

Therefore,

$$Y(s) = \frac{1}{2}\left(\frac{1}{s} - \frac{s+1}{(s+1)^2 + 1} + \frac{1}{(s+1)^2 + 1}\right)$$

and then

$$y(t) = \mathfrak{L}^{-1}(Y(s)) = \frac{1}{2}(1 - e^{-t}\cos t + e^{-t}\sin t).$$