QUIZ 5

Solve the following initial value problem

$$y'' + 4y = g(t), \quad y(0) = 0, \quad y'(0) = 0.$$

where

$$g(t) = \begin{cases} t & \text{if } 0 \le t < 2 \\ 2 & \text{if } t \ge 2. \end{cases}$$

Solutions:

First $g(t) = t + u_2(t)(2-t) = t - u_2(t)(t-2)$. Applying Laplace transform to the question and set $Y(s) = \mathfrak{L}(y)$, we get

$$(s^2+4)Y(s) = \mathfrak{L}(t) - \mathfrak{L}(u_2(t)(t-2)).$$

By the formula that $\mathfrak{L}(u_c(t)f(t-c)) = e^{-cs}F(s)$ with $F(s) = \mathfrak{L}(f(t))$, we get

$$(s^2+4)Y(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2}.$$

So

$$Y(s) = \frac{1}{s^2(s^2+4)} - \frac{e^{-2s}}{s^2(s^2+4)}.$$

Since $\frac{1}{s^2(s^2+4)} = \frac{1}{4}(\frac{1}{s^2} - \frac{1}{s^2+4})$, we see

$$\mathfrak{L}^{-1}(\frac{1}{s^2(s^2+4)}) = \frac{1}{4}(t - \frac{1}{2}\sin(2t)).$$

Finally, using the formula $\mathfrak{L}^{-1}(e^{-cs}F(s)) = u_c(t)f(t-c)$, we find

$$y = \mathfrak{L}^{-1}(Y(s)) = \frac{1}{4}\left(t - \frac{1}{2}\sin(2t)\right) - \frac{1}{4}u_2(t)\left(t - 2 - \frac{1}{2}\sin(2(t-2))\right).$$