# Math 266, Practice Midterm 1

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Name: ________________________________________

This exam consists of 7 pages including this front page.

## Ground Rules

1. Calculator is not allowed.
2. Show your work for every problem unless otherwise stated (partial credits are available).
3. You may use one 4-by-6 index card, both sides.

**Part I: Multiple Choice (5 points) each.** For each of the following questions circle the letter of the correct answer from among the choices given. (No partial credit.)

1. Which of the following is a 2nd order *non-linear* differential equation.

   (a) $ty'' = \sin t y' - \frac{1}{t}y + t^2$.
   (b) $y'' = 3y' + 4y - 6$.
   (c) $(y')^2 = 6ty - e^t$.
   (d) $y'' = 3e^t(y')^2 + \sin y + 5e^t$.
   (e) $y'y = 6e^t$.

2. Initially a tank holds 50 gallons of pure water. A salt solution containing $\frac{1}{3}$ lb of salt per gallon runs into the tank at the rate of 5 gallons per minute. The well mixed solution runs out of the tank at a rate of 2 gallons per minute. Let $x(t)$ be the amount of salt in the tank at time $t$. Find a differential equation satisfied by $x(t)$.

   (a) $x' = \frac{5}{3} - \frac{2x}{50+3t}$.
   (b) $x' = \frac{5}{2} - \frac{3x}{50+3t}$.
   (c) $x' = \frac{5}{3} - \frac{3x}{50+2t}$.
   (d) $x' = \frac{5}{2} - \frac{2x}{50+3t}$.
   (e) $x' = \frac{5}{3} - \frac{2x}{50+2t}$.

3. Which of the following forms a fundamental solutions for the equation

$$y'' - 2y' + 6 = 0$$

   (a) $\{e^t \cos 2t, e^t \sin 2t\}$
   (b) $\{e^{2t} \cos t, e^{2t} \sin t\}$
   (c) $\{e^{-2t} \cos t e^{-2t} \sin t\}$
   (d) $\{e^{-t} \cos 2t, e^{-t} \sin 2t\}$
   (e) None of the above.

4. Consider the homogeneous differential equation

$$\frac{dy}{dx} = \frac{x^2 + 3y^2}{6xy}$$

   By replacing $v = y/x$, the equations becomes

(a) $v' = \frac{1+3v^2}{6v}$

(b) $xv' = \frac{1+3v^2}{6v} - v$

(c) $v' = \frac{1+3v^2}{6v} - v$

(d) $xv' = \frac{1+3v^2}{6v}$

(e) $v' = \frac{1+3v^2}{6v} + v$

**Answer Key:** (d), (a), (e), (b).

**Part II: Written answer questions.**

5. Find the particular solution to the following initial value problem (15 points)

$$4y'' - 8y' + 3y = 0,\ y(0) = 2,\ y'(0) = \frac{1}{2}.$$

*Solution:* The characteristic polynomial is $f(r) = 4r^2 - 8r + 3$. The roots of $f(r) = 0$ are $r_1 = \frac{1}{2}$ and $r_2 = \frac{3}{2}$. Therefore the general solution of the equation is

$$y = C_1 e^{\frac{1}{2}t} + C_2 e^{\frac{3}{2}t}$$

So $y' = \frac{C_1}{2} e^{\frac{1}{2}t} + \frac{3C_2}{2} e^{\frac{3}{2}t}$. Plug in the initial value, we get

$$C_1 + C_2 = 2,\ \ \frac{C_1}{2} + \frac{3C_2}{2} = \frac{1}{2}.$$

So we get $C_1 = \frac{5}{2}$ and $C_2 = -\frac{1}{2}$ and then

$$y = \frac{5}{2} e^{\frac{1}{2}t} - \frac{1}{2} e^{\frac{3}{2}t}.$$

6. Solve each of the following initial value problems, and give the interval on which the solution is valid.

   (a) $ty' + 4y = t^{-2}e^t,\ y(1) = 2.$ (10 points)

   *Solution:* It is equivalent to solve $y' + \frac{4}{t}y = t^{-3}e^t,\ y(1) = 2$. The integrating factor $\mu(t) = \exp(\int \frac{4}{t}dt) = t^4$. Then

$$y(t) = \frac{1}{t^4}(\int t^4 t^{-3} e^t dt + C)$$

   So $y(t) = \frac{1}{t^4}((t-1)e^t + C)$. Since $y(1) = 2$. We see that $C = 2$ and

$$y(t) = \frac{1}{t^4}((t-1)e^t + 2).$$

   Hence $y(t)$ is valid if $t \neq 0$.

   (b) $y' = \frac{4x}{1+2y},\ y(1) = -1.$ (10 points)

   *Solution:* This is separable equation and we have $(1+2y)dy = 4xdx$. Integral both sides, we get

$$y + y^2 = 2x^2 + C$$

   Since $y(1) = -1$, we see that $C = -2$. Hence we have $y + y^2 = 2(x^2 - 1)$. Then

$$y^2 + y + \frac{1}{4} = 2x^2 - \frac{7}{4}$$

   So $(y + \frac{1}{2})^2 = 2(x^2 - \frac{7}{8})$ and

$$y = -\frac{1}{2} \pm \sqrt{2(x^2 - \frac{7}{8})}$$

   Hence $y$ is valid if $|x| \geq \sqrt{\frac{7}{8}}$. But the initial value problem rule out the possibility of $2y + 1 = 0$, that is, $y$ can not be $\frac{1}{2}$. So $x^2$ can not be $\frac{7}{8}$. Finally, $y$ is valid for $|x| > \sqrt{\frac{7}{8}}$.

7. Consider the differential equation

$$\frac{dy}{dt} = (y-3)(5-y)y. \tag{1}$$

(a) Find equilibrium solutions. (5 points)

*Solution:*

$$y = 0,\ 3,\ 5.$$

(b) Sketch the direction field of the equation (1). (5 points)

(c) Decide the stability of each equilibrium solutions. (5 points)

*Solution:* $y = 5, 0$ are stable. $y = 3$ is unstable.

8. Consider the differential equation $xy^2 + bx^2y + (x^3 + yx^2)y' = 0$.

(a) Find the value of $b$ such that the equation is exact. (7 points)
*Solution:* We see that $M(x,y) = xy^2 + bx^2y$ and $N(x,y) = x^3 + yx^2$. The equation is exact if and only if

$$\frac{\partial M}{\partial y} = 2xy + bx^2 = \frac{\partial N}{\partial x} = 3x^2 + 2xy.$$

So $b = 3$ the equation is exact.

(b) Solve the differential equation for the value of $b$ given in question (a). (8 points)

*Solution:* First $F(x,y) = \int M(x,y)dx = \int (xy^2+3x^2y)dx = \frac{1}{2}x^2y^2+x^3y$. Then

$$N(x,y) - \frac{\partial F}{\partial y} = x^3 + yx^2 - (x^2y + x^3) = 0.$$

So we have $h'(y) = 0$ and then we can set $h(y) = 0$. The solution of the equation is

$$\Psi(x,y) = F(x,y) + h(y) = \frac{1}{2}x^2y^2 + x^3y = C.$$

9. A ball with a mass of $\frac{1}{2}$kg is throw upwards with an initial velocity of 10 m/sec from the roof of a building 40m high. Assume there is air resistance of $\frac{|v|}{25}$ where the velocity v is measured in m/sec. Assuming constant acceleration dues to gravity of g = 9.8m/sec.

   (a) When the ball stops to rise? (5 points)

   *Solution:* We have the deceleration $v' = -9.8 - (\frac{|v|}{25})/0.5$. Since $v \geq 0$, we get the initial value problem

$$v' = -9.8 - \frac{2v}{25}, \ v(0) = 40.$$

   We easily see $a = -\frac{2}{2}5$ and $b = 9.8$. So

$$v(t) == -(4.9 \times 25) + (40 + (4.9 \times 25))e^{-\frac{2t}{25}}.$$

   That is, we have $v(t) = -122.5 + 162.5e^{-\frac{2t}{25}}$. Solve $v(t) = 0$, we find the time when ball stop to rise is $t_0 = \frac{25}{2}\ln(\frac{162.5}{122.5})$.

   (b) Find the maximal height above the ground that the ball reaches. (5 points)

   *Solution:* Let $s(t)$ be the distance that ball traveled at the time $t$. We have $s(t) = \int v(t)dt$. In particular, the maximal height is

$$\begin{aligned} s(t_0) &= \int_0^{t_0} (-122.5 + 162.5e^{-\frac{2t}{25}})dt \\ &= -122.5t - (12.5 \times 162.5)e^{-\frac{2t}{25}}|_0^{t_0} \\ &= 12.5 \times 162.5(1 - e^{-\frac{2t_0}{25}}) - 122.5. \end{aligned}$$

   (c) Find the time that the ball hits the ground. (5 points)

   *Solution:* Now we have differential equation $v' = -9.8 + (\frac{|v|}{25})/0.5$. But $v \leq 0$ all the time, we still have the equation $v' = -9.8 - \frac{2v}{25}$. So the solution of previous steps applies. That is, $v(t) = -122.5 + 162.5e^{-\frac{2t}{25}}$ and

$$s(t) = \int_0^t v(s)ds = -122.5t + (12.5 \times 162.5)(1 - e^{-\frac{2t}{25}}).$$

   Solve $s(t_1) = -40$. Then $t_1$ is the time that ball hits the ground.