Math 266, Midterm Exam 1

February 19th 2016

Name: ______________________________________

Ground Rules:

1. Calculator is NOT allowed.
2. Show your work for every problem unless otherwise stated (partial credits are available).
3. You may use one 4-by-6 index card, both sides.

Part I: Multiple Choice (5 points) each. For each of the following questions circle the letter of the correct answer from among the choices given. (No partial credit.)

1. Which of the following is a 2nd order linear differential equation.

   (a) $ty'' = \sin t(y')^2 - \frac{1}{t}y + t^2$.

   (b) $y'' = 3y' + 4e^y - 6$.

   (c) $(y')^2 = 6ty - e^t$.

   (d) $(\cos t)y'' = 3e^t y' + y + 5e^t$.

   (e) $y'y = 6e^t$.

2. Which of the following is the general solution for the equation

$$y'' + 4y' + 6y = 0$$

   (a) $C_1 e^{-2t}\cos 2t + C_2 e^{-2t}\sin 2t$

   (b) $C_1 e^{-2t}\cos\sqrt{2}t + C_2 e^{-2t}\sin\sqrt{2}t$

   (c) $C_1 e^{-2t} + C_2 e^{2t}$

   (d) $C_1 e^{(-2+\sqrt{2})t} + C_2 e^{(-2+\sqrt{2})t}$

   (e) None of the above.

3. Consider the *homogeneous* equation

$$\frac{dy}{dx} = \frac{x^2 - 3y^2}{6xy}.$$

   Set $v = \frac{y}{x}$. Then which is following the equation between v and x:

   (a) $\frac{dv}{dx} = \frac{x^2 - v^2}{6xv}$

   (b) $\frac{dv}{dx} = \frac{1-3v^2}{6v}$

   (c) $x\frac{dv}{dx} = \frac{1-3v^2}{6v}$

   (d) $x\frac{dv}{dx} + v = \frac{1-3v^2}{6v}$

   (e) None of the above

4. Consider the initial value problem $y' = 2t - y$ and $y(0) = 1$. Use Euler's method with step size $h = 0.5$ to compute y_2 to approximate $y(1)$. Then $y_2 =$

   (a) 0.7
   (b) 0.75
   (c) 0.5
   (d) 1
   (e) $\frac{1}{e}$

   Answer Keys: **d, b, d, b**

Part II: Written answer questions. Details of explanation are required. Partial credits are available.

5. Consider the initial value problem:

$$y'' - y' + 0.25y = 0, \ y(0) = 2, \ y'(0) = a$$

   (a) Find the solution as a function of a (8 points).

   Solutions: The characteristic polynomial $f(r) = r^2 - r + 0.25 = (r - \frac{1}{2})^2$. So it has repeated roots $r_1 = r_2 = \frac{1}{2}$. Hence $e^{\frac{t}{2}}$ and $te^{\frac{t}{2}}$ forms a fundamental set of solutions. That is

$$y = C_1 e^{\frac{t}{2}} + C_2 t e^{\frac{t}{2}}.$$

   Now use the initial conditions $y(0) = 2$ and $y'(0) = 2$ and note that $y' = \frac{1}{2}C_1 e^{\frac{t}{2}} + C_2 e^{\frac{t}{2}} + \frac{1}{2}C_2 t e^{\frac{t}{2}}$. We get equations $C_1 = 2$ and $\frac{1}{2}C_1 + C_2 = a$. Hence $C_1 = 2$ and $C_1 = a - 1$. That is

$$y = 2e^{\frac{t}{2}} + (a-1)te^{\frac{t}{2}} = e^{\frac{t}{2}}((a-1)t + 2).$$

   (b) Determine the critical value of a that separates solutions that grow positively from those that eventually grow negatively (4 points).

   Solutions: Note that $e^{\frac{t}{2}}$ always increases and is positive. So $y(t)$ depends on $(a-1)t + 2$ to grows positively or negatively. It is obviously that if $(a-1) > 0$ then $(a-1)t + 2$ grows positively, and if $a - 1 < 0$ then $(a-1)t + 2$ grows negatively. Hence the required critical value $a = 1$.

6. Show that $y_1 = t^2$, $y_2 = t^{-1}$ forms a fundamental set of solutions for the equation

$$t^2 y'' - 2y = 0, \quad t > 0. \quad \text{(10 points)}$$

Solutions: We first check that $y_1 = t^2$ and $y_2 = t^{-1}$ solutions. We have

$$y_1' = 2t,\ y_1'' = 2; \quad y_2' = -t^{-2},\ y_2'' = 2t^{-3}.$$

So plug in y_i, y_i' and y_i'' for $i = 1, 2$ to the equation $t^2 y'' - 2y = 0$. We easily see y_1 and y_2 are solutions.

Now we check that y_1, y_2 forms a a fundamental set of solutions by computing their Wronskian

$$W(t^2, t^{-1}) = \begin{vmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{vmatrix} = -1 - 2 = -3 \neq 0.$$

So $y_1 = t^2$, $y_2 = t^{-1}$ forms a fundamental set of solutions.

7. Solve the following initial value problem, and give the interval on which the solution is valid. (13 points)

$$y' = \frac{-x+1}{1+y}, \ y(1) = -2.$$

Solutions: This is a separable equation and we get

$$(1+y)dy = (1-x)dx$$

and then

$$\int (1+y)dy = \int (1-x)dx.$$

We get

$$y + \frac{y^2}{2} = x - \frac{x^2}{2} + C.$$

Or by multiplying 2, we have

$$2y + y^2 = 2x - x^2 + C.$$

Since $y(1) = -2$, we get $C = -1$ and then we can rewrite as

$$1 + 2y + y^2 = 2x - x^2.$$

Or

$$(y+1)^2 = 2x - x^2.$$

So

$$y = -1 \pm \sqrt{2x - x^2}.$$

Since $y(1) = -2$, indeed we have

$$y = -1 - \sqrt{2x - x^2}.$$

To see the interval so that the solution is valid, note that the square root forces $2x - x^2 = x(2-x) \geq 0$, which is equivalent to that $0 \leq x \leq 2$. Finally, x can not be $0, 2$: This would mean that $y = -1$ and then $y + 1$ will be zero but $y + 1$ is in the denominator in the equation. So the interval in which the solution is valid is $x \in (0, 2)$.

8. Consider the differential equation

$$\frac{dy}{dt} = (y-1)^2(2-y)y, \quad t \geq 0. \tag{1}$$

(a) Find equilibrium solutions. (5 points)

Solutions: $y = 0, 1, 2$.

(b) Sketch the direction field of the equation. (5 points)

(c) Decide the stability of each equilibrium solutions. (5 points)

Solutions: $y = 0$ is unstable, $y = 1$ is semi-stable and $y = 2$ is stable.

9. Consider the differential equation

$$ye^{2xy} + x + b(xe^{2xy} + 1)\frac{dy}{dx} = 0.$$

(a) Find the value of b such that the equation is exact. (6 points)

Solution:
We have $M(x, y) = ye^{2xy} + x$ and $N(x, y) = b(xe^{2xy} + 1)$. So the equation is exact if only if

$$\frac{\partial M}{\partial y} = e^{2xy} + 2xye^{2xy} = \frac{\partial N}{\partial x} = b(e^{2xy} + 2xye^{2xy}).$$

So $b = 1$.

(b) Solve the differential equation for the value of b given in question (a). (9 points)

Solutions: We have

$$F(x, y) = \int M(x, y)dx = \int ye^{2xy} + xdx = \frac{e^{2xy}}{2} + \frac{x^2}{2}$$

and

$$h'(y) = N(x, y) - \frac{\partial F}{\partial y} = xe^{2xy} + 1 - xe^{2xy} = 1$$

So $h(y) = y$. Finally the general solution is

$$\Psi(x, y) = F(x, y) + h(y) = \frac{e^{2xy}}{2} + \frac{x^2}{2} + y = C.$$

10. A tank with a capacity of 500 gal originally contains 200 gal of water with 100 lb of salt in solution. Water containing 1 lb salt per gallon is entering at a rate of 3 gal/min, and the mixture is allowed to flow out of the tank at a rate of 2 gal/min. Let $x(t)$ be the amount of salt in the tank at time t.

(a) Find $x(t)$ at any time prior to the instant when the solution begins to overflow. (5 points)

Solutions: Let $x(t)$ be the amount of salt in the tank at the time t and $V(t)$ is the volume of the water in tank. Note that $V(t) = 200 + t$. So the differential equation we need is

$$x'(t) = 3 - \frac{2x(t)}{V(t)} = 3 - \frac{2x(t)}{200+t} \quad \text{and} \quad x(0) = 100.$$

This is a linear equation and with the integrate factor $\mu(t) = \exp(\int \frac{2}{200+t}) = (t+200)^2$. So the general solution is

$$x(t) = \frac{1}{(t+200)^2}\left(\int 3(t+200)^2 dt + C\right) = \frac{(t+200)^3 + C}{(t+200)^2}.$$

Plug in $x(0) = 100$, we get

$$x(t) = \frac{(t+200)^3 - 4 \times 10^6}{(t+200)^2}.$$

(b) Find the concentration of salt in the tank when it is on the point of overflowing. (5 points)

Solutions: First note the time of overflow is the time t_0 such that $V(t_0) = 200 + t_0 = 500$. So $t_0 = 300$. Then the concentration at t_0 is

$$\frac{x(t_0)}{V(t_0)} = \frac{x(300)}{V(300)} = \frac{500^3 - 4 \times 10^6}{500^2 \times 500} = \frac{5^3 - 4}{5^3} = 0.968.$$

(c) Assume that the tank has infinite capacity, what is the theoretic concentration when the time goes to infinity? (5 points)

Solutions: The concentration $C(t)$ at t is $\frac{x(t)}{V(t)}$. If the tank has infinite capacity then $V(t) = 200 + t$. So we get

$$C(t) = \frac{x(t)}{t+200} = \frac{(t+200)^3 - 4 \times 10^6}{(t+200)^2 \times (t+200)} = 1 - \frac{4 \times 10^6}{(200+t)^3}.$$

So we see that

$$\lim_{t \to +\infty} C(t) = 1.$$