Math 266 Midterm Exam 2

March 21st 2016

Name: ______________________________

Ground Rules

1. Calculator is NOT allowed.
2. Show your work for every problem unless otherwise stated (partial credits are available).
3. You may use one 4-by-6 index card, both sides.

Part I: Multiple Choice (5 points) each. For each of the following questions circle the letter of the correct answer from among the choices given. (No partial credit.)

1. To use the method of undetermined coefficients to find a particular solution of the differential equation

$$y^{(4)} + y'' = 4\cos t + 2 + e^t,$$

which of the following forms of $Y(t)$ should we try?

(a) $Y(t) = A\cos t + B + Ce^t$

(b) $Y(t) = At\cos t + Bt^2 + Cte^t$

(c) $Y(t) = At\cos t + Bt\sin t + C + Dte^t$

(d) $Y(t) = At\cos t + Bt\sin t + Ct^2 + De^t$

(e) $Y(t) = A\cos t + B\sin t + Ct^2 + De^t$

2. Suppose that $3\cos(2t) - 4\sin(2t) = R\cos(2t - \delta)$. Then what are R and δ ?

(a) $R = 5,\ \delta = \pi - \tan^{-1}(\frac{4}{3})$.

(b) $R = 5,\ \delta = -\tan^{-1}(\frac{4}{3})$.

(c) $R = 5,\ \delta = \pi - \tan^{-1}(\frac{3}{4})$.

(d) $R = 5,\ \delta = -\tan^{-1}(\frac{3}{4})$.

(e) Not enough information to determine R and δ.

3. Consider the differential equation

$$y^{(5)} + 3y^{(4)} + 2y^{(3)} + 6y'' + y' + 3y = 0$$

Assume that the characteristic polynomial $f(r) = (r+3)(r^2+1)^2$, then the general solution has the form

(a) $y(t) = C_1 e^{-3t} + C_2 \cos t + C_3 \sin t$
(b) $y(t) = C_1 e^{-3t} + C_2 \cos t + C_3 \sin t + C_4 \cos 2t + C_5 \sin 2t$
(c) $y(t) = C_1 e^{-3t} + C_2 \cos t + C_3 \sin t + C_4 t \cos t + C_5 t \sin t$
(d) $y(t) = C_1 e^{-3t} + C_2 e^{t} + C_3 t e^{t} + C_4 e^{-t} + C_5 t e^{-t}$
(e) $y(t) = C_1 e^{-3t} + (C_2 t + C_3) \sin t + C_4 \cos 2t + C_5 \sin 2t$

4. Suppose the following the initial value problem describes the motion of a certain spring-mass system:

$$4u''(t) + u'(t) + 2u(t) = 0,\ u(0) = 1,\ u'(0) = 0.$$

Then what we can conclude for $\lim_{t\to+\infty} u(t)$?

(a) $\lim_{t\to+\infty} u(t)$ does not exists.
(b) $\lim_{t\to+\infty} = +\infty$.
(c) $\lim_{t\to+\infty} u(t) = -\infty$.
(d) $\lim_{t\to+\infty} u(t) = 0$.
(e) $\lim_{t\to+\infty} u(t) = 1$

5. Suppose the $y(t)$ is the solution of the following initial value problem:

$$y'' - y' = e^t + 2t,\ y(0) = 0,\ y'(0) = 0.$$

Then $y(1) =$?

(a) $2e - 5$
(b) $5e - 3$.
(c) $2e - 2$.
(d) $3e - 5$
(e) None of the above.

Answer Key: (d), (b), (c), (d), (e). The answer to No.5 should be $y(1) = 2e - 4$. Indeed, $y(t) = e^t - 1 + te^t - (t^2 + 2t)$.

Part II: Written answer questions

6. Assume that $y_1(x) = e^x$ is a solution of the following differential equation

$$(x-1)y'' - xy' + y = 0, \quad x > 1$$

Find the general solution of the above homogeneous equation. (15 points)

Solutions: First we write the equation to the following shape

$$y'' - \frac{x}{x-1}y' + \frac{y}{x-1} = 0$$

Let $y_2(x) = v(x)y_1(x)$. Then method of reduction of order tells us that v satisfies the following equation

$$e^x v'' + (2e^x - \frac{x}{x-1}e^x)v' = 0.$$

Or equivalently: $v'' + \frac{x-2}{x-1}v' = 0$. This is the first order ODE of v' and it is separable equation. We get

$$\int \frac{dv'}{v'} = -\int \frac{x-2}{x-1}dx = \int -1 + \frac{1}{x-1}dx = -x + \ln(x-1).$$

So $\ln v' = -x + \ln(x-1)$ and $v' = e^{-x}(x-1)$. Then

$$v = \int e^{-x}(x-1)dx = -xe^{-x}$$

Then $y_2(x) = vy_1 = -xe^{-x}e^x = -x$. The general solution of the equation is given by

$$y = C_1 e^x + C_2 x$$

Finally, we check $W(y_1, y_2) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x(1-x) \neq 0$ to conclude that $\{e^x, x\}$ does form a fundamental set of solutions of the equation.

7. Consider the differential equation

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3,\ t > 0. \tag{1}$$

(a) Show that $y_1(t) = t$ and $y_2(t) = te^t$ forms a fundamental set of solutions for the homogeneous equation $t^2y'' - t(t+2)y' + (t+2)y = 0$. (10 points)

Solutions: It is easy to check that both y_1 and y_2 are solutions of the equation. The Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} t & te^t \\ 1 & e^t(t+1) \end{vmatrix} = e^t t(t+1) - te^t = t^2e^t \neq 0$$

(b) Find a particular solution of equation (1). (10 points)

Solutions: We first write the equation to the following form:

$$y'' - \frac{t+2}{t}y' + \frac{t+2}{t^2}y = 2t.$$

In particular, $g(t) = 2t$. According to the method of variation of parameters, we have a particular solution given by

$$y = -y_1 \int \frac{y_2 g}{W(y_1, y_2)} dt + y_2 \int \frac{y_1 g}{W(y_1, y_2)} dt.$$

So

$$y = -t \int \frac{te^t \cdot 2t}{t^2e^t} dt + te^t \int \frac{t \cdot 2t}{t^2e^t} dt = -t \int 2dt + te^t \int 2e^{-t} dt.$$

Hence $y = -2t^2 - 2t$.

8. A mass weighting 16 lb stretches a spring 2 feet. The mass is displaced 6 in. in the positive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $2\cos 3t$ lb. Assume that the gravity constant is $g = 32ft/sec^2$.

   (a) Formulate the initial value problem describing the motion of the mass. (6 points)

   Solution: We have $mg = 16$ and $L = 2$. Hence $k = mg/L = 8$ and $m = 16/32 = 1/2$. Note the $u(0)$ should be 6 in $= 1/2$ ft. Now we get the initial value problem:

$$\frac{1}{2}u'' + 8u = 2\cos 3t,\ u(0) = \frac{1}{2},\ u'(0) = 0.$$

   (b) Find the solution of the initial value problem. (8 points)

   Solution: The above equation is equivalent to $u'' + 16u = 4\cos 3t$. It is easy to see the characteristic polynomial of the homogenous equation is $r^2 + 4^2$. Then the general solution for the homogenous equation is $C_1 \cos 4t + C_2 \sin 4t$. To find a particular solution, we assume that $Y(t) = A\cos 3t + B\sin 3t$ with $A,\ B$ undetermined coefficients. Then we have $B = 0$ and $A = \frac{4}{7}$. So the general solution is

$$u(t) = C_1 \cos 4t + C_2 \sin 4t + \frac{4}{7}\cos 3t.$$

   Now by the initial value condition, we get

$$C_1 + \frac{4}{7} = \frac{1}{2},\ \text{and } C_2 = 0.$$

   Then we find the solution of this initial value problem.

$$u(t) = -\frac{1}{14}\cos 4t + \frac{4}{7}\cos 3t.$$

   (c) If the given external force is replaced by a force $4\cos\omega t$ of frequency ω and the mass is attached to viscous damper with a damping constant 2 lb-sec/ft., find the value of ω for which resonance occurs. (6 points)

   Solution: By the formula, we know resonance occurs when $\omega = \omega_{\max}$ satisfies the following equation

$$\omega_{\max}^2 = \omega_0^2(1 - \frac{\gamma^2}{2mk}).$$

Here we have $\omega_0 = \sqrt{k/m} = 4$, $\gamma = 2$, $k = 8$ and $m = \frac{1}{2}$. So

$$\omega_{\max} = 4\sqrt{1 - \frac{4}{2 \cdot \frac{1}{2} \cdot 8}} = \frac{4}{\sqrt{2}}.$$

9. Consider the following function

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ a, & t = 1 \\ t, & \;\; t > 1. \end{cases}$$

(a) Show that $f(t)$ is a piecewise continuous function (5 points).

Solutions:

First $f(t)$ are obviously continuous over $(0,1)$ and $(1,+\infty)$. At $t=1$, we have

$$\lim_{t\to 1-} f(t) = \lim_{t\to 1-} 1 = 1 \text{ and } \lim_{t\to 1+} f(t) = \lim_{t\to 1+} t = 1$$

Both these limits are finite. So $f(t)$ is piecewise continuous.

(b) Find Laplace transform $F(s) = \mathcal{L}(f(t))$ (15 points).

Solutions:

$$\begin{aligned} \mathcal{L}\{f(t)\} = \int_0^{+\infty} f(t)e^{-st}dt &= \int_0^1 f(t)e^{-st}dt + \int_1^{+\infty} f(t)e^{-st}dt \\ &= \int_0^1 1 \cdot e^{-st} + \int_1^{+\infty} te^{-st}dt. \end{aligned}$$

Note that $f(1) = \alpha$ does not affect the integral so we have $\int_1^{+\infty} f(t)e^{-st}dt = \int_1^{+\infty} te^{-st}dt$. Now

$$\int_0^1 e^{-st}dt = \left.\frac{e^{-st}}{-s}\right|_0^1 = \frac{1}{s} - \frac{e^{-s}}{s}$$

We use integral by part for the following integral

$$\int_1^{+\infty} te^{-st}dt = \int_1^{+\infty} t\left(\frac{e^{-st}}{-s}\right)' dt = \left.\frac{te^{-st}}{-s}\right|_1^{+\infty} - \int_1^{\infty} \frac{e^{-st}}{-s}dt = \left.\frac{te^{-st}}{-s}\right|_1^{+\infty} - \left.\frac{e^{-st}}{s^2}\right|_1^{+\infty}.$$

Since

$$\lim_{t \to +\infty} te^{-st} = \lim_{t \to +\infty} e^{-st} = 0,$$

we have

$$\int_1^{+\infty} te^{-st} dt = \frac{e^{-s}}{s} + \frac{e^{-s}}{s^2}.$$

So finally,

$$\mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{e^{-s}}{s^2}.$$