

Math 353 Practice Final Exam

Name: _____

This exam consists of 12 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

<i>Score</i>		
1	16	
2	24	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
<i>Total</i>	100	

Notations: \mathbb{R} denotes the set of real number; F is always a field, for example, $F = \mathbb{R}$; $M_{m \times n}(F)$ denotes the set of $m \times n$ -matrices with entries in F ; $F^n = M_{n \times 1}(F)$ denotes the set of n -column vectors; $P_n(F)$ denotes the set of polynomials with coefficients in F and the most degree n , that is,

$$P_n(F) = \{f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in F, \forall i\}.$$

V is always a finite dimensional vector space over F and T is always a linear operator $T : V \rightarrow V$. A^* always denote complex conjugate and transpose of A . For an eigenvalue λ of A , E_λ denotes the λ -eigenspace and K_λ denote the generalized eigenspace.

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. (2 points each)

- (a) There is a matrix A with an eigenvalue λ such that the algebraic multiplicity of λ is 5 and $\dim K_\lambda = 4$.
- (b) Let v_1, \dots, v_m are nonzero generalized eigenvectors of A with eigenvalues $\lambda_1, \dots, \lambda_m$. Suppose that $\lambda_1, \dots, \lambda_m$ are distinct then v_1, \dots, v_m are linearly independent.
- (c) For an matrix A with all real entries, then all eigenvalues of A are real numbers.
- (d) Suppose the Jordan canonical form A has one Jordan block with size > 1 . Then A is diagonalizable.
- (e) Suppose $W = \text{Span}\{v_1, \dots, v_n\}$. Then $u \in W^\perp$ if and only if u and v_i are orthogonal for all $i = 1, \dots, n$.
- (f) Let A is be an invertible matrix . Then singular values of A are the same as eigenvalues of A .
- (g) If A is a square matrix then $B = A - A^*$ is normal.
- (h) If A is unitary then $\det(A) = \pm 1$.

	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
Answer	F	T	F	F	T	F	T	F

2. Multiple Choice. (3 points each)

- (i) Let A be an $m \times n$ -matrix. Consider the system of linear equations $AX = b$, which of the following statement is always true:
- (a) Suppose $m > n$ then rank of augmented matrix $(A|b)$ can not larger than m .
 - (b) Suppose $m \leq n$ then $(A^*A)X = A^*b$ always has a unique solution.
 - (c) Suppose A has full rank then $AX = b$ always has a solution.
 - (d) If $AX = b$ has a solution then $AX = 0$ has unique solution.
 - (e) If $AX = b$ has a unique solution then A has to be invertible.

The correct answer is (a).

- (ii) Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 3 \\ -2 \\ 2 \end{pmatrix}$. Which vectors from e_1, e_2, e_3, e_4 should be added to v_1, v_2 to form a basis of \mathbb{R}^4 ?

- (a) e_1, e_2
- (b) e_2, e_3
- (c) e_2, e_4
- (d) e_1, e_4
- (e) such vectors do not exists.

The correct answer is (d).

- (iii) Let V be the real vector space of continuous function over $[-1, 1]$ with the inner product $\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx$. Which of the following set is orthonormal?
- (a) $1, x, x^2$
 - (b) $1, e^x$
 - (c) $1, x, x^2 - \frac{1}{3}$
 - (d) $\sin x, \cos x$

(e) $\frac{1}{\sqrt{2}}, \frac{\sqrt{3}x}{\sqrt{2}}$.

The correct answer is (e).

(iv) Let

$$A = \begin{pmatrix} 0 & 0 & 0 & a \\ -1 & 0 & 0 & b \\ 0 & -1 & 0 & c \\ 0 & 0 & -1 & d \end{pmatrix}$$

Suppose 0 is an eigenvalue of A with algebraic multiplicity 2. Then which of the following statement is correct?

- (a) $a = 0, b \neq 0$
- (b) $a = b = 0$ and $c \neq 0$
- (c) $a = c = 0$ but $b \neq 0$.
- (d) $a = d = 0$ but $b \neq 0$.
- (e) $b = c = 0$ but $a \neq 0$.

The correct answer is (b).

(v) Suppose $T : V \rightarrow V$ be a linear operator with characteristic polynomial $f(t) = t^3 - t^2$. Which of the following statement is always correct?

- (a) T is an isomorphism.
- (b) T can not be diagonalizable.
- (c) For any $v \in V, T^3(v) = T^2(v)$.
- (d) Such T is unique.
- (e) It is possible that T is a unitary operator.

The correct answer is (c).

(vi) Which of the following statement is NOT equivalent that $A \in M_{n \times n}(\mathbb{C})$ is invertible?

- (a) Columns of A are linearly independent.

- (b) A is normal.
- (c) All eigenvalues of A are nonzero.
- (d) A has n positive singular values.
- (e) The linear system $AX = b$ has unique solution.

The correct answer is (b).

(vii) Let $A = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0.4 & 0.6 \end{pmatrix}$. Which of the following statement is NOT correct?

- (a) A is a regular transition matrix.
- (b) A is diagonalizable.
- (c) All eigenvalues of A are real numbers.
- (d) $\lim_{m \rightarrow \infty} A^m$ exists.

(e) $\lim_{m \rightarrow \infty} A^m = \begin{pmatrix} \frac{1}{3} & 0.5 & 0.5 \\ \frac{1}{3} & 0.5 & 1 \\ \frac{1}{3} & 0 & 0.5 \end{pmatrix}$.

The correct answer is (e).

(viii) Consider the linear system $\begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. Consider the least

square solution $\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$ of the above system then $\hat{y} =$

- (a) $\frac{1}{5}(\frac{3}{2}a - \frac{3}{2}b - \frac{c}{2} + \frac{d}{2})$.
- (b) $\frac{a}{2} + \frac{b}{2} + \frac{c}{2} + \frac{d}{2}$.
- (c) $(\frac{3}{2}a - \frac{3}{2}b - \frac{c}{2} + \frac{d}{2})$.
- (d) $2a - b + d$.
- (e) None of the above answers are correct.

The correct answer is (a).

3. Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 \\ -2 & -2 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$.

- (a) Find Jordan canonical form J of A . (5 points)
 (b) Find the invertible matrix S so that $A = SJS^{-1}$. (5 points)

Solutions: By cofactor expansion, it is not hard to compute the characteristic polynomial $f_A(t) = t^2(t-1)^2$. Hence we get two distinct eigenvalue $\lambda_1 = 0$ and $\lambda_2 = 1$ both with algebraic multiplicity 2. Now let us compute the K_λ for each λ_i .

For $\lambda_1 = 0$, we have to solve $AX = 0$. We easily see that the eigenspace E_{λ_1} has dimension 1, generated by $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$. Now $A^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -3 & -3 & 3 & 3 \\ 2 & 2 & -2 & -2 \end{pmatrix}$.

By solving $A^2X = 0$, we get basis v_1, v_2 of K_{λ_1} where $v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$. Now we

build cycle $\{Av_2, v_2\}$, we get $Av_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} = v_1$.

Now for $\lambda_2 = 1$, we have $(A - I) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ -2 & -2 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$. The eigenspace

has dimension 1 which is spanned by $v_3 := \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$. We have

$$(A - I)^2 = \begin{pmatrix} -1 & -2 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $N(A-I) = K_{\lambda_2}$ is spanned by v_3 and e_4 . We build a cycle $(A-I)e_4, e_4$. We easily calculate that $Ae_4 = v_3$.

In summary, the Jordan canonical form of A is $\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. And

$$S = (v_1, v_2, v_3, e_4) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

4. Compute singular value decomposition of $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$. (10 points)

Solutions: To obtain the SVD of A , we first decompose A^*A .

We first have $A^*A = 2 \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}$. It is not hard to calculate the characteristic polynomial is $f(t) = -t^2(t-6)$. For $\lambda_1 = 6$, we easily compute that the eigenspace has a basis $v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$. For $\lambda_2 = 0$, the eigenspace

is defined by equation $x - y + z = 0$. So we get two basis $v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. But v_2 and v_3 are not orthogonal. By Gram-Schmidt as in the

next problem, we can replace v_3 by $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$. By replacing v_i by $v_i/\|v_i\|$. We obtained orthonormal eigenvectors as basis, and hence

$$V = (v_1, v_2, v_3) = \left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right)$$

Therefore $A^*A = V\Lambda V^*$ where Λ has 6, 0, 0 on the main diagonal. So the singular value only has $\sigma_1 = \sqrt{6}$.

Now $u_1 = \frac{1}{\sigma_1}v_1 = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Now it suffices to select u_2 such that u_1, u_2 forms an orthonormal basis of \mathbb{R}^2 . We can select $u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. Finally, we get a SVD of A :

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^*.$$

5. Let $W \subset \mathbb{R}_3$ be the subspace spanned by $w = (1, 1, 1)$. Let W^\perp be the orthogonal complement of W . Let $v = (1, 0, 1)$.

1. Find an orthonormal basis of W^\perp . (4 points)
2. Find the projection of v to W^\perp . (3 points)
3. Find the $\min_{w \in W^\perp} \|w - v\|$. (3 points)

Solutions : It is easy to check that $W^\perp := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + y + z = 0 \right\}$. We can

pick two basis vectors of W^\perp to be $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. But v_1 and v_2 is not orthogonal. By Gram-Schmidt, we set $w_1 = v_1$ and

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{pmatrix}.$$

Replace w_i by $w_i/\|w_i\|$, we obtain an orthonormal basis of W^\perp : $u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Now we can use the formula of projection

$$\text{Proj}_{W^\perp} v = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

So the projection is

$$0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Then

$$\min_{w \in W^\perp} \|w - v\| = \|v - \text{Proj}_{W^\perp} v\| = \left\| \frac{2}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\| = \frac{2}{\sqrt{3}}.$$

6. Let A be an $n \times n$ -matrix with real entries. Suppose A is skew-symmetric, that is, $A^T = -A$.
- (a) Show that if n is odd then $\det(A) = 0$ (3 points)
 - (b) Show that A is always diagonalizable. (3 points)
 - (c) Show that eigenvalues of A are either 0 or purely imaginary, that is, $\lambda = bi$ for $b \in \mathbb{R}$. (4 points)

proof

- (a) Since $A^T = -A$, we have $\det(A^T) = \det(-A) = (-1)^n \det(A)$. Note that n is odd, $(-1)^n = -1$. So $\det(A) = \det(A^T) = -\det(A)$. That is, $\det(A) = 0$.
- (b) Since A is real matrix and $A^T = -A$, we have $A^*A = A^T A = -AA = A(-A) = AA^*$. That is A is normal. So A is diagonalizable.
- (c) Since A is normal, A admits spectral decomposition $A = U\Lambda U^*$ where U is a unitary matrix and Λ is a diagonal matrix with eigenvalue λ_i on the main diagonal. Since A is real matrix, we have $A^T = A^* = (U\Lambda U^*)^* = (U^*)^* \Lambda^* U^* = U\Lambda^* U^*$. But $A^T = -A = -U\Lambda U^*$. So $U\Lambda^* U^* = -U\Lambda U^*$. Since U is invertible, we have $\Lambda^* = -\Lambda$. That is, $\bar{\lambda}_i = \lambda_i$ for all i . Then λ_i is either 0 or imaginary.

7. Suppose A and B are square matrices and $AB = BA$.

- (a) If v is a generalized eigenvector of A with eigenvalue λ . So is Bv . (5 points)
- (b) Suppose that all eigenvalues of A are distinct. Show that there exists an invertible matrix S so that $A = S\Lambda_1S^{-1}$ and $B = S\Lambda_2S^{-1}$ with Λ_1, Λ_2 being diagonal matrices. (5 points)

Proof:

- (a) Let K_λ be the generalized eigenspace of A with the eigenvalue λ and $v \in K_\lambda$. So $(A - \lambda I)^p v = \vec{0}$ for some $p > 0$. Now we check that $(A - \lambda I)^p B = B(A - \lambda I)^p$. Indeed, since $AB = BA$,

$$\begin{aligned} (A - \lambda I)^p B &= (A - \lambda I)^{p-1} (A - \lambda I) B = (A - \lambda I)^{p-1} (AB - \lambda B) = \\ &= (A - \lambda I)^{p-1} (BA - B\lambda I) = (A - \lambda I)^{p-1} B(A - \lambda I). \end{aligned}$$

Repeat the above steps, we have $(A - \lambda I)^p B = B(A - \lambda I)^p$. Then $(A - \lambda I)^p Bv = B(A - \lambda I)^p v = B\vec{0} = \vec{0}$. That is, $Bv \in K_\lambda$.

- (b) Since eigenvalues of A are distinct, A is diagonalizable. So there exists an invertible matrix S such that $A = S\Lambda_1S^{-1}$ where Λ_1 is a diagonal matrix with distinct eigenvalues λ_i in the main diagonal. Since $AB = BA$, we have $S\Lambda_1S^{-1}B = BS\Lambda_1S^{-1}$. Note that S is invertible, this is equivalent to that $\Lambda_1(S^{-1}BS) = (S^{-1}BS)\Lambda_1$. Now we claim that $C = S^{-1}BS = (c_{ij})$ is necessarily a diagonal matrix. In fact Λ_1C is equivalent to times λ_i to the i -th row of C , where $C\Lambda_1$ is equivalent to times λ_j to j -th column of C . Then $\Lambda_1C = C\Lambda_1$ means that $\lambda_i c_{ij} = \lambda_j c_{ij}$. Since all λ_i are distinct, we conclude that $c_{ij} = 0$ unless $i = j$. That is $S^{-1}BS = C = \Lambda_2$ is a diagonal matrix. So $B = S\Lambda_2S^{-1}$.

8. Let $A \in M_{m \times n}(\mathbb{C})$. Show the following:

1. $\text{rank}(AA^*) = \text{rank}(A)$. (5 points)
2. If λ is an eigenvalue of AA^* then $\lambda \geq 0$. (5 points).

Proof:

By singular value decomposition, $A = U\Sigma V^*$ where U and V are unitary matrices, and Σ has singular values $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > 0$ on the main diagonal, and $r = \text{rank}(A)$. Then

$$AA^* = U\Sigma V^*(U\Sigma V^*)^* = U\Sigma V^*V\Sigma^*U^* = U\Sigma\Sigma^*U^*.$$

It is easy to check that $\Lambda := \Sigma\Sigma^*$ is an $m \times m$ -matrix with σ_i^2 on the main diagonal for $i = 1, \dots, r$. In particular, AA^* is similar to $\Lambda := \Sigma\Sigma^*$ and hence they share the same eigenvalues and the same rank (note that U is invertible so $\text{rank}(AA^*) = \text{rank}(U\Lambda U^*) = \text{rank}(\Lambda)$). Hence AA^* 's eigenvalues λ_i are either σ_i^2 or 0. Hence $\lambda_i \geq 0$. Since $\sigma_i^2 > 0$ has exact $r = \text{rank}(A)$ many on the main diagonal of Λ , $\text{rank}(AA^*) = \text{rank}(\Lambda) = r = \text{rank}(A)$.