Math 353 Practice Final Exam

Name: __________________________________________

This exam consists of 12 pages including this front page.

Ground Rules

1. No calculator is allowed.

2. Show your work for every problem unless otherwise stated.

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Notations: \( \mathbb{R} \) denotes the set of real number; \( F \) is always a field, for example, \( F = \mathbb{R} \); \( M_{m \times n}(F) \) denotes the set of \( m \times n \)-matrices with entries in \( F \); \( F^n = M_{n \times 1}(F) \) denotes the set of \( n \)-column vectors; \( P_n(F) \) denotes the set of polynomials with coefficients in \( F \) and the most degree \( n \), that is,

\[
P_n(F) = \{ f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \ a_i \in F, \ \forall i \}.
\]

\( V \) is always a finite dimensional vector space over \( F \) and \( T \) is always a linear operator \( T : V \to V \). \( A^* \) always denote complex conjugate and transpose of \( A \). For an eigenvalue \( \lambda \) of \( A \), \( E_\lambda \) denotes the \( \lambda \)-eigenspace.

1. The following are true/false questions. You don’t have to justify your answers. Just write down either T or F in the table below. (2 points each)

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(a) There is a matrix \( A \) with an eigenvalue \( \lambda \) such that the multiplicity of \( \lambda \) is 5 and \( \dim E_\lambda = 4 \).

(b) Let \( V \) be an inner product space with \( \alpha \) an ordered basis and \( T : V \to V \) a linear operator. If \( A = [T]_\alpha \) then \( A^* = [T^*]_\alpha \).

(c) For an matrix \( A \) with all real entries, then all eigenvalues of \( A \) are real numbers.

(d) Let \( A \in M_{n \times n}(\mathbb{C}) \) be a square matrix. The rank of \( A \) is the same as the number of nonzero eigenvalues.

(e) Suppose \( W = \text{Span}\{v_1, \ldots, v_n\} \). Then \( u \in W^\perp \) if and only if \( u \) and \( v_i \) are orthogonal for all \( i = 1, \ldots, n \).

(f) Let \( A \) be an invertible matrix. Then singular values of \( A \) are the same as eigenvalues of \( A \).

(g) If \( A \) is a square matrix then \( B = A - A^* \) is normal.

(h) If \( A \) is unitary then \( \det(A) = \pm 1 \).
2. Multiple Choice. (3 points each)

(i) Let $A$ be an $m \times n$-matrix. Consider the system of linear equations $AX = b$, which of the following statement is always true:

(a) Suppose $m > n$ then rank of augmented matrix $(A|b)$ can not larger than $m$.
(b) Suppose $m \leq n$ then $(A^*A)X = A^*b$ always has a unique solution.
(c) Suppose $A$ has full rank then $AX = b$ always has a solution.
(d) If $AX = b$ has a solution then $AX = 0$ has unique solution.
(e) If $AX = b$ has a unique solution then $A$ has to be invertible.

The correct answer is (a).

(ii) Let $v_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 3 \\ -2 \\ 2 \end{pmatrix}$. Which vectors from $e_1, e_2, e_3, e_4$ should be added to $v_1, v_2$ to form a basis of $\mathbb{R}^4$?

(a) $e_1, e_2$
(b) $e_2, e_3$
(c) $e_2, e_4$
(d) $e_1, e_4$
(e) such vectors do not exists.

The correct answer is (d).

(iii) Let $V$ be the real vector space of continuous function over $[-1, 1]$ with the inner product $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$. Which of the following set is orthonormal?

(a) $1, x, x^2$
(b) $1, e^x$
(c) $1, x, x^2 - \frac{1}{3}$
(d) $\sin x, \cos x$
The correct answer is (e).

(iv) Let

\[
A = \begin{pmatrix}
0 & 0 & 0 & a \\
-1 & 0 & 0 & b \\
0 & -1 & 0 & c \\
0 & 0 & -1 & d
\end{pmatrix}
\]

Suppose 0 is an eigenvalue of \( A \) with multiplicity 2. Then which of the following statement is correct?

(a) \( a = 0, b \neq 0 \)
(b) \( a = b = 0 \) and \( c \neq 0 \)
(c) \( a = c = 0 \) but \( b \neq 0 \).
(d) \( a = d = 0 \) but \( b \neq 0 \).
(e) \( b = c = 0 \) but \( a \neq 0 \).

The correct answer is (b).

(v) Suppose \( T : V \to V \) be a linear operator with characteristic polynomial \( f(t) = t^3 - t \). Which of the following statement is always correct?

(a) \( T \) is an isomorphism.
(b) \( T \) can not be diagonalizable.
(c) For any \( v \in V \), \( T^2(v) = T(v) \).
(d) Such \( T \) is unique.
(e) It is possible that \( T \) is a unitary operator.

The correct answer is (c).

(vi) Which of the following statement is NOT equivalent that \( A \in M_{n \times n}(\mathbb{C}) \) is invertible?

(a) Columns of \( A \) are linearly independent.
(b) $A$ is normal.
(c) All eigenvalues of $A$ are nonzero.
(d) $A$ has $n$ positive singular values.
(e) The linear system $AX = b$ has unique solution.

The correct answer is (b).

(vii) Let $\lambda \in \mathbb{C}$ and $m > 1$. Consider the following $m \times m$-matrix $J_\lambda = \begin{pmatrix} \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$ Which of the following statement is correct?

(a) $A$ is a normal matrix.
(b) $A$ is diagonalizable.
(c) $\dim E_\lambda > 1$
(d) $\lim_{m \to \infty} A^m$ always exists for any $\lambda$.
(e) $A$ is not a unitary matrix.

The correct answer is (e).

(viii) Consider the linear system $\begin{pmatrix} 1 & 2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. Consider the least square solution $\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix}$ of the above system then $\hat{y} =$

(a) $\frac{1}{5}(\frac{3}{2}a - \frac{3}{2}b - \frac{c}{2} + \frac{d}{2})$.
(b) $\frac{a}{2} + \frac{b}{2} + \frac{c}{2} + \frac{d}{2}$.
(c) $(\frac{3}{2}a - \frac{3}{2}b - \frac{c}{2} + \frac{d}{2})$.
(d) $2a - b + d$.
(e) None of the above answers are correct.

The correct answer is (a).
3. Let $A = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.4 & 0.2 & 0.4 \\ 0 & 0.4 & 0.6 \end{pmatrix}$.

(a) Is $A$ a regular transition matrix? explain. (5 points)

(b) Find $\lim_{m \to \infty} A^m$. (5 points)

Solutions:

(a) It is clear that $A$ is a transition matrix because summation of each column is 1 and all entries are non-negative. It is easy to see that each entry of $A^2$ is positive. So $A$ is regular.

(b) Since $A$ is a regular transition matrix, we know 1 is an eigenvalue. Solve $(A - I)X = \vec{0}$, we find a eigenvector $X = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. By dividing 3, $\frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is the eigenvector of $A$ with eigenvalue 1 and it is also a probability vector. So

$$\lim_{m \to \infty} A^m = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$
4. Compute singular value decomposition of \( A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix} \). (10 points)

_Solutions:_ To obtain the SVD of \( A \), we first decompose \( A^*A \).

We first have \( A^*A = 2 \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \). It is not hard to calculate the characteristic polynomial is \( f(t) = -t^2(t-6) \). For \( \lambda_1 = 6 \), we easily compute that the eigenspace has a basis \( v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \). For \( \lambda_2 = 0 \), the eigenspace is defined by equation \( x - y + z = 0 \). So we get two basis \( v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \) and \( v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \). But \( v_2 \) and \( v_3 \) are not orthogonal. By Gram-Schmidt as in the next problem, we can replace \( v_3 \) by \( \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \). By replacing \( v_i \) by \( v_i/\|v_i\| \). We obtained orthonormal eigenvectors as basis, and hence

\[
V = (v_1, v_2, v_3) = \left( \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \right)
\]

Therefore \( A^*A = V \Lambda V^* \) where \( \Lambda \) has \( 6, 0, 0 \) on the main diagonal. So the singular value only has \( \sigma_1 = \sqrt{6} \).

Now \( u_1 = \frac{1}{\sigma_1} Av_1 = \left( \frac{1}{1} \right) \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \). Now it suffices to select \( u_2 \) such that \( u_1, u_2 \) forms an orthonormal basis of \( \mathbb{R}^2 \). We can select \( u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \). Finally, we get a SVD of \( A \):

\[
A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} V^*.
\]
5. Let $W \subset \mathbb{R}^3$ be the subspace spanned by $w = (1, 1, 1)$. Let $W^\perp$ be the orthogonal complement of $W$. Let $v = (1, 0, 1)$.

1. Find an orthonormal basis of $W^\perp$. (4 points)
2. Find the projection of $v$ to $W^\perp$. (3 points)
3. Find the $\min_{w \in W^\perp} ||w - v||$. (3 points)

Solutions: It is easy to check that $W^\perp := \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} | x + y + z + 0 \}$. We can pick two basis vector of $W^\perp$ to be $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. But $v_1$ and $v_2$ is not orthogonal. By Gram-Schmidt, we set $w_1 = v_1$ and

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$ 

Replace $w_i$ by $w_i/||w_i||$, we obtain an orthonormal basis of $W^\perp$: $u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.

Now we can use the formula of projection

$$\text{Proj}_{W^\perp} v = \frac{\langle v, u_1 \rangle}{\langle u_1, u_1 \rangle} u_1 + \frac{\langle v, u_2 \rangle}{\langle u_2, u_2 \rangle} u_2$$

So the projection is

$$0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Then

$$\min_{w \in W^\perp} ||w - v|| = ||v - \text{Proj}_{W^\perp} v|| = ||\frac{2}{3} \begin{pmatrix} 1 \\ 1 \end{pmatrix}|| = \frac{2}{\sqrt{3}}.$$
6. Let \( A \) be an \( n \times n \)-matrix with real entries. Suppose \( A \) is skew-symmetric, that is, \( A^T = -A \).

(a) Show that if \( n \) is odd then \( \det(A) = 0 \) (3 points)

(b) Show that \( A \) is always diagonalizable. (3 points)

(c) Show that eigenvalues of \( A \) are either 0 or purely imaginary, that is, \( \lambda = bi \) for \( b \in \mathbb{R} \). (4 points)

**proof**

(a) Since \( A^T = -A \), we have \( \det(A^T) = \det(-A) = (-1)^n \det(A) \). Note that \( n \) is odd, \( (-1)^n = -1 \). So \( \det(A) = \det(A^T) = -\det(A) \). That is, \( \det(A) = 0 \).

(b) Since \( A \) is real matrix and \( A^T = -A \), we have \( A^*A = A^TA = -AA = A(-A) = AA^* \). That is \( A \) is normal. So \( A \) is diagonalizable.

(c) Since \( A \) is normal, \( A \) admits spectral decomposition \( A = U\Lambda U^* \) where \( U \) is a unitary matrix and \( \Lambda \) is a diagonal matrix with eigenvalue \( \lambda_i \) on the main diagonal. Since \( A \) is real matrix, we have \( A^T = A^* = (U\Lambda U^*)^* = (U^*)^*\Lambda^*U^* = U\Lambda^*U^* \). But \( A^T = -A = -U\Lambda U^* \). So \( U\Lambda^*U^* = -U\Lambda U^* \). Since \( U \) is invertible, we have \( \Lambda^* = -\Lambda \). That is, \( \lambda_i = \lambda_i \) for all \( i \). Then \( \lambda_i \) is either 0 or imaginary.
7. Suppose $A$ and $B$ are square matrices and $AB = BA$.

(a) If $v$ is an eigenvector of $A$ with eigenvalue $\lambda$ then $Bv$ is in $\lambda$-eigenspace of $A$. (5 points)

(b) Suppose that all eigenvalues of $A$ are distinct. Show that there exists an invertible matrix $S$ so that $A = S\Lambda_1 S^{-1}$ and $B = S\Lambda_2 S^{-1}$ with $\Lambda_1, \Lambda_2$ being diagonal matrices. (5 points)

Proof:

(a) Let $E_\lambda$ be the eigenspace of $A$ with the eigenvalue $\lambda$. Since $v \in E_\lambda$ then $Av = \lambda v$ and

$$A(Bv) = (BAv) = B\lambda v = \lambda Bv.$$

Therefore, $Bv \in E_\lambda$.

(b) Since eigenvalues of $A$ are distinct, $A$ is diagonalizable. So there exists an invertible matrix $S$ such that $A = S\Lambda_1 S^{-1}$ where $\Lambda_1$ is a diagonal matrix with distinct eigenvalues $\lambda_i$ in the main diagonal. Since $AB = BA$, we have $SA_1 S^{-1} B = BSA_1 S^{-1}$. Note that $S$ is invertible, this is equivalent to that $\Lambda_1(S^{-1}BS) = (S^{-1}BS)\Lambda_1$. Now we claim that $C = S^{-1}BS = (c_{ij})$ is necessarily a diagonal matrix. In fact $\Lambda_1 C$ is equivalent to times $\lambda_i$ to the $i$-th row of $C$, where $CA_1$ is equivalent to times $\lambda_j$ to $j$-th column of $C$. Then $\Lambda_1 C = CA_1$ means that $\lambda_i c_{ij} = \lambda_j c_{ij}$. Since all $\lambda_i$ are distinct, we conclude that $c_{ij} = 0$ unless $i = j$. That is $S^{-1}BS = C = \Lambda_2$ is a diagonal matrix. So $B = S\Lambda_2 S^{-1}$. 
8. Let \( A \in M_{m \times n}(\mathbb{C}) \). Show the following:

1. \( \text{rank}(AA^*) = \text{rank}(A) \). (5 points)
2. If \( \lambda \) is an eigenvalue of \( AA^* \) then \( \lambda \geq 0 \). (5 points)

Proof:

By singular value decomposition, \( A = U\Sigma V^* \) where \( U \) and \( V \) are unitary matrices, and \( \Sigma \) has singular values \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \) on the main diagonal, and \( r = \text{rank}(A) \). Then

\[
AA^* = U\Sigma V^*(U\Sigma V^*)^* = U\Sigma V^*\Sigma V^*U^* = U\Sigma \Sigma^* U^*.
\]

It is easy to check that \( \Lambda := \Sigma \Sigma^* \) is an \( m \times m \)-matrix with \( \sigma_i^2 \) on the main diagonal for \( i = 1, \ldots, r \). In particular, \( AA^* \) is similar to \( \Lambda := \Sigma \Sigma^* \) and hence and they share the same eigenvalues and the same rank (note that \( U \) is invertible so \( \text{rank}(AA^*) = \text{rank}(U \Lambda U^*) = \text{rank}(\Lambda) \)). Hence \( AA^* \)'s eigenvalues \( \lambda_i \) are either \( \sigma_i^2 \) or 0. Hence \( \lambda_i \geq 0 \). Since \( \sigma_i^2 > 0 \) has exact \( r = \text{rank}(A) \) many on the main diagonal of \( \Lambda \), \( \text{rank}(AA^*) = \text{rank}(\Lambda) = r = \text{rank}(A) \).