Math 353, Practice Midterm 1

Name: ________________________________

This exam consists of 8 pages including this front page.

Ground Rules

1. You may take 3 by 5 cards two sides as note.
2. No calculator is allowed.
3. Show your work for every problem unless otherwise stated.

<table>
<thead>
<tr>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>Total</td>
</tr>
</tbody>
</table>
1. The following are true/false questions. You don’t have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(b)</th>
<th>(c)</th>
<th>(d)</th>
<th>(e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Answer</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

(a) Let $W_1, W_2$ be subspaces of a vector space $V$, then $W_1 \cap W_2$ is a subspace of $V$.

(b) Let $V$ be a vector space of dimension $n$. Then any set of $m$ vectors with $m < n$ is linearly independent.

(c) Let $T : V \to W$ be a linear transformation of vector spaces $V$ and $W$. If $S \subset V$ is a basis of $V$, then $T(S)$ spans $R(T)$.

(d) Let $\beta$ be a basis of $V$ and $T : V \to V$ a linear transformation. Write $A = [T]_\beta$. Then $T$ is an isomorphism if and only if $N(A) = \{0\}$.

(e) Let $A \in M_{n \times n}(F)$ be a square matrix. Then the linear transformation $L_A : F^n \to F^n$ is an isomorphism if and only if $A$ is invertible.
2. Multiple Choice, \( A, B, C, X, b \) are always matrices here:

(i) Which of the following statement is correct.

(a) Any set of 4 vectors in \( F^4 \) is a basis of \( F^4 \)
(b) Any set of 4 vectors in \( F^3 \) is linearly dependent.
(c) Any set of 2 vectors in \( F^3 \) is linearly independent.
(d) Any set of 5 vectors in \( F^4 \) must span \( F^4 \)
(e) Any linearly independent subset of \( F^3 \) is a basis of \( F^3 \)

The correct answer is (b).

(ii) Which of the given subsets of \( \mathbb{R}^3 \) is a subspace?

(a) The set of all vectors of the form \( (a, b, c) \) such that \( 2a + b = c \)
(b) The set of all vectors of the form \( (a, b, c) \) such that \( a + b < c \)
(c) The set of all vectors of the form \( (a, b, c) \) such that \( 2a + b = 1 \)
(d) The set of unit sphere.
(e) The set of all vectors of the form \( (a, a^2, a^3) \).

The correct answer is (a).

(iii) Which of the following map is a linear transformation

(a) \( L : \mathbb{R}^3 \to \mathbb{R}^2 \) by \( L\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = \begin{pmatrix} x + y \\ y + 1 \end{pmatrix} \).
(b) \( T : P_3(\mathbb{R}) \to \mathbb{R} \) by \( T(f(x)) = \int_0^1 f(x)^2 dx \)
(c) \( S : P_3(\mathbb{R}) \to \mathbb{R} \) by \( S(f(x)) = f'(3) + 2 \).
(d) \( L : M_{2 \times 2}(F) \to M_{2 \times 2}(F) \) by \( L(X) = AXA \) where \( A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \).
(e) \( L : \mathbb{R} \to \mathbb{R} \) by \( L(x) = e^x \).

The correct answer is (d).

(iv) Let \( T : V \to W \) be a linear transformation. Assume that \( \dim V = \dim W = n \) and let \( \alpha \subset V \) and \( \beta \subset W \) be bases. Then which of the following statement is NOT equivalent to that \( T \) is an isomorphism

(a) \( T \) is one-to-one.
(b) $T$ is onto.
(c) The matrix $[T]_\beta^\alpha$ is invertible.
(d) $T$ sends any linearly independent set of $V$ to linearly independent set in $W$.
(e) $T$ sends any linearly dependent set of $V$ to linearly dependent set in $W$.

The correct answer is (e).

(v) Let $A$ be a $3 \times 5$-matrix of real numbers and $A \neq 0$. Which of the following statement is FALSE?
(a) $L_A$ define by $L_A(x) = Ax$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^5$.
(b) rank($A$) $\leq 3$
(c) If $\alpha$ and $\beta$ are the standard bases for $\mathbb{R}^5$ and $\mathbb{R}^3$ respectively then $[L_A]_\beta^\alpha = A$.
(d) Nullity($A$) $\geq 2$.
(e) $L_A$ can not be one-to-one.

The correct answer is (a).
3. Let \( p_1 = 1 + x, \ p_2 = x + 2x^2 - x^3, \ p_3 = 1 + 2x + 2x^2 - x^3, \ p_4 = 1 + 2x + 3x^2 + x^3 \) be polynomials in \( P_3(\mathbb{R}) \). Find a basis for \( \text{Span}\{p_1, p_2, p_3, p_4\} \subset P_3(\mathbb{R}) \).

Solutions: Consider equation of vectors \( a_1p_1 + a_2p_2 + a_3p_3 + a_4p_4 = 0 \) with \( a_i \) being unknowns. compare the coefficients of each degree, we arrive the following system of linear equations

\[
\begin{pmatrix}
1 & 0 & 1 & 1 \\
1 & 1 & 2 & 2 \\
0 & 2 & 2 & 3 \\
0 & -1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Use elementary operation to simplify the equation, we arrive

\[
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3 \\
a_4
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Then the equation has a solution \( a_4 = 0, \ a_1 = 1, \ a_2 = 1, \ a_3 = -1 \) So it is clear that \( p_1, p_2, p_3 \) are linear dependent. If we remove \( p_3 \) or equivalently set \( a_3 = 0 \). We see that this forces \( a_1 = a_2 = a_4 = 0 \). So \( p_1, p_2, p_4 \) are linearly independent and hence a basis for \( \text{Span}\{p_1, p_2, p_3, p_4\} \).
4. Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation such that $T\begin{pmatrix}1 \\ 1\end{pmatrix} = \begin{pmatrix}2 \\ 1\end{pmatrix}$ and $T\begin{pmatrix}1 \\ 2\end{pmatrix} = \begin{pmatrix}2 \\ 1\end{pmatrix}$.

(a) Find $T\begin{pmatrix}2 \\ 1\end{pmatrix}$.

(b) Find the standard matrix representing $T$.

(c) Find nullity of $T$ and the rank of $T$.

(d) Given a basis $\beta = \{\begin{pmatrix}2 \\ 1\end{pmatrix}, \begin{pmatrix}1 \\ 0\end{pmatrix}\}$. Find matrix $[T]_\beta$.

**Solutions:**

(a) Solve equation $\begin{pmatrix}2 \\ 1\end{pmatrix} = x \begin{pmatrix}1 \\ 1\end{pmatrix} + y \begin{pmatrix}1 \\ 2\end{pmatrix}$. We see $x = 3$, $y = -1$. Then

$$T\begin{pmatrix}2 \\ 1\end{pmatrix} = 3T\begin{pmatrix}1 \\ 1\end{pmatrix} - T\begin{pmatrix}1 \\ 2\end{pmatrix} = \begin{pmatrix}4 \\ 2\end{pmatrix}.$$

(b) We find $\begin{pmatrix}1 \\ 0\end{pmatrix} = 2\begin{pmatrix}1 \\ 1\end{pmatrix} - \begin{pmatrix}1 \\ 2\end{pmatrix}$ and $\begin{pmatrix}0 \\ 1\end{pmatrix} = -\begin{pmatrix}1 \\ 1\end{pmatrix} + \begin{pmatrix}1 \\ 2\end{pmatrix}$. So similarly as the above, we have

$$T\begin{pmatrix}1 \\ 0\end{pmatrix} = \begin{pmatrix}2 \\ 1\end{pmatrix}, \quad T\begin{pmatrix}0 \\ 1\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}.$$

So the standard matrix representing $T$ is just $A = \begin{pmatrix}2 & 0 \\ 1 & 0\end{pmatrix}$.

(c) Since rows in $A$ are just multiple to each other, row space of $A$ only has 1 linearly dependent vector. So $\text{rank}(A) = \text{rank}(T) = 1$ and the nullity of $T$ is $2 - \text{rank}(A) = 1$.

(d) Let $Q = \begin{pmatrix}2 & 1 \\ 1 & 0\end{pmatrix}$. Then $[T]_\beta = Q^{-1}AQ = \begin{pmatrix}2 & 1 \\ 4 & 2\end{pmatrix}$. 

6
5. Let \( v_1, \ldots, v_n \in F^n \) and \( A = (v_1, \ldots, v_n) \) be the \( n \times n \)-matrix. Show that \( \{v_1, \ldots, v_n\} \) is a basis of \( F^n \) if and only if \( A \) is an invertible matrix via the following steps:

(a) Show that there exists a linear transformation \( T : F^n \rightarrow F^n \) so that \( T(e_i) = v_i \) for all \( i \). Here \( \alpha := \{e_1, \ldots, e_n\} \) is the standard basis of \( F^n \).

(b) Show that \( \left[ T \right]_\alpha = A \).

(c) Show that if \( \{v_1, \ldots, v_n\} \) is a basis of \( F^n \) then there exists a linear transformation \( U : F^n \rightarrow F^n \) so that \( U(v_i) = e_i \) for all \( i \)

(d) Show that \( \{v_1, \ldots, v_n\} \) is a basis of \( F^n \) if and only if \( A \) is an invertible matrix.

Proof:

(a) Since \( \{e_1, \ldots, e_n\} \) is the standard basis of \( F^n \). By Theorem 2.6, there exists unique linear transformation \( T : F^n \rightarrow F^n \) such that \( T(e_i) = v_i \) for all \( i = 1, \ldots, n \).

(b) Since \( \alpha \) is the standard basis of \( F^n \), we always have \( v = [v]_\alpha \) for any \( v \in F^n \). So

\[
\left[ T \right]_\alpha = ([T(e_1)]_\alpha, \ldots, [T(e_n)]_\alpha) = ([v_1]_\alpha, \ldots, [v_n]_\alpha) = (v_1, \ldots, v_n) = A.
\]

(c) If \( v_1, \ldots, v_n \) is a basis of \( F^n \) then we can applies Theorem 2.6 to \( \{v_i\} \) and \( \{e_i\} \) again, which implies that there exists a unique linear transformation \( U : F^n \rightarrow F^n \) so that \( U(v_i) = e_i \) for all \( i = 1, \ldots, n \).

(d) Applies Theorem 2.18 to \( T \), we see \( T \) is an isomorphism if and only if \( A = [T]_\alpha \) is invertible. Now if \( v_1, \ldots, v_n \) is a basis, then (c) constructed a linear transformation \( U : F^n \rightarrow F^n \). Note that \( UT(e_i) = I_{F^n}(e_i) \) for all \( i \). Using Theorem 2.6 again, \( UT \) is necessarily \( I_{F^n} \). Similarly, we see that \( TU = I_{F^n} \). So \( T \) is invertible and hence \( A = [T]_\alpha \) is invertible. Conversely, if \( A \) is invertible then \( T \) is invertible, which means it is one-to-one and onto. In particular, by Theorem 2.2, \( \{v_i = T(e_i), i = 1, \ldots, n\} \) spans \( F^n \). Since \( \{v_1, \ldots, v_n\} \) has \( n \) vectors, this implies that \( \{v_1, \ldots, v_n\} \) is a basis of \( F^n \).
6. Let $V$, $W$ be vector spaces over field $F$. Let $v_1, \ldots, v_n \in V$ be linearly independent vectors.

(a) Show that given $w_1, \ldots, w_n \in W$ then there exists a linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for all $i = 1, \ldots, n$.

(b) Could we drop the assumption of linear independence for so that the above statement is still true? Why or why not?

(c) Is $T$ necessarily unique, why or why not?

Solutions:

(a) Proof: Since $v_i$ are linearly independent, then one can always extend $v_i$, say $\{v_1, \ldots, v_n, v_{n+1}, \ldots, v_m\}$, to be a basis of $V$. Now extend $w_i$ to $m$-vectors $w_1, \ldots, w_n, w_{n+1}, \ldots, w_m$. For example, $w_{n+1} = \cdots = w_m = 0$. Now by Theorem 2.6, there exists a unique linear transformation $T : V \to W$ so that $T(v_i) = w_i$ for all $i = 1, \ldots, m$.

(b) No. We can not drop the assumption that $v_i$ are linearly independent. For example, let $V = W = \mathbb{R}$, $v_1 = v_2 = 1$, but $w_1 = 0$ and $w_2 = 2$. It is not possible to define a linear transformation $T : \mathbb{R} \to \mathbb{R}$ so that $T(v_i) = w_i$.

(c) Contrary to Theorem 2.6, $T$ here in general is not unique as $v_i$ may not form a basis. For example, let $V = W = \mathbb{R}^2$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $w_1 = v_1$. We see $I_V(v_1) = w_1$. On the other hand, consider linear transformation $T\left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x \\ 0 \end{pmatrix}$. We still have $T(v_1) = w_1$. 

8