# Math 353, Midterm 1

Name: ______________________________

This exam consists of 9 pages including this front page.

**Ground Rules**

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.
3. You may use one 3 by 5 index card, both sides.

| *Score* | | |
|---|---|---|
| 1 | 15 | |
| 2 | 20 | |
| 3 | 15 | |
| 4 | 15 | |
| 5 | 20 | |
| 6 | 15 | |
| *Total* | 100 | |

**Notations:** $\mathbb{R}$ denotes the set of real number; $F$ is always a field, for example, $F = \mathbb{R}$; $M_{m\times n}(F)$ denotes the set of $m \times n$-matrices with entries in $F$; $F^n = M_{n\times 1}(F)$ denotes the set of $n$-column vectors; $P_n(F)$ denotes the set of polynomials with coefficients in $F$ and the most degree $n$, that is,

$$P_n(F) = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in F, \ \forall i\}.$$

Given $A \in M_{m\times n}(F)$, then the linear transformation $L_A : F^n \to F^m$ is defined via $L_A(x) = Ax$ for any $x \in F^n$.

**1.** The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. (3 points each)

(a) Let $W_1, W_2$ be subspaces of a vector space $V$, then $W_1 \cup W_2$ is a subspace of $V$.

(b) Let $T : V \to W$ be a linear transformation then

$$\text{nullity}(T) + \text{rank}(T) = \dim W.$$

(c) Let $U$, $T : V \to V$ be two linear transformations from $V$ to $V$. Then $UT$ is an isomorphism if and only if both $U$ and $T$ are isomorphisms.

(d) Let $A \in M_{n\times n}(F)$ be a square matrix. Consider the linear transformation $L_A : F^n \to F^n$ via $L_A(x) = Ax$. Then for any basis $\beta$ of $F^n$, we have $[L_A]_\beta = A$.

(e) If rows of a matrix $A$ are linearly independent then so are the columns.

| | (a) | (b) | (c) | (d) | (e) |
|---|---|---|---|---|---|
| Answer | F | F | T | F | F |

**2.** Multiple Choice. (4 points each)

(i) Which of the following vector space is isomorphic to $P_3(F)$ (which is the space of polynomials at most degree 3)?

(a) The column space of an invertible $3 \times 3$-matrix $A$.

(b) The subspace $W$ of $M_{2\times 2}(F)$ defined by

$$W := \{A \in M_{2\times 2}(F) | A^T = A\}.$$

(c) The space $\mathcal{L}(F^2) := \{\text{all linear transformations } T : F^2 \to F^2\}$.

(d) The subspace of $W$ of $F_5$ defined by

$$W = \{(a_1, a_2, a_3, a_4, a_5) \in F_5 | a_1 + a_2 = 0,\ \ a_3 = 2a_4\}$$

(e) None of the above.

The correct answer is (C)

(ii) Suppose that $A$ is an $m \times n$ matrix with entries in $\mathbb{R}$. Which of the following statement is NOT correct?

(a) $\text{rank}(A) = \dim \text{Col}(A)$ and where $\text{Col}(A)$ denotes the column space of $A$.

(b) $\text{rank}(A) = \text{rank}(AE)$ for any $n \times n$-invertible matrix $E$.

(c) $\text{rank}(A) = \dim \text{Row}(A)$ and where $\text{Row}(A)$ denotes the row space of $A$.

(d) $\text{rank}(A) = \min\{m, n\}$.

(e) If $\text{rank}(A) < n$ then columns of $A$ are linearly dependent.

The correct answer is (D)

(iii) Suppose $V$ is a vector space over $F$ with $\dim V = 5$. Which of the following statement is NOT correct.

(a) Any set of 6 vectors in $V$ spans $V$.
(b) Any set of 6 vectors in $V$ is linearly dependent.
(c) $V$ is isomorphic to $F^5$.
(d) If a set $S$ of vectors in $V$ is linearly independent then there are at most 5 vectors in $S$.
(e) Suppose $S = \{v_1, v_2, v_3, v_4, v_5\}$ spans $V$ then $S$ is a basis of $V$.

The correct answer is (a)

(iv) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation so that $T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\0\\2\end{pmatrix}$ and $T\begin{pmatrix}2\\3\end{pmatrix} = \begin{pmatrix}1\\-1\\4\end{pmatrix}$. Which of the following statement is FALSE?

(a) $T$ is one-to-one.
(b) Such a linear transformation $T$ is unique.
(c) The standard matrix representing $T$ (i.e., the matrix representing $T$ for the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$) is $\begin{pmatrix}1 & -1 & 2\\2 & -2 & 4\end{pmatrix}$.
(d) $\text{rank}(T) = 2$.
(e) $T\begin{pmatrix}-1\\-2\end{pmatrix} = \begin{pmatrix}0\\1\\-2\end{pmatrix}$.

The correct answer is (C)

(v) Let $A \in M_{n\times n}(F)$ be a square matrix. Which of the following statement is NOT equivalent to the statement that $A$ is invertible.

(a) The linear transformation $L_A : F^n \to F^n$ given by $L_A(x) = Ax$ is an isomorphism.

(b) $\text{rank}(A) = n$

(c) $L_A$ sends *any* linearly dependent set of $F^n$ to linearly dependent set in $F^n$.

(d) $A$ is a product of elementary matrices.

(e) Rows of $A$ are linearly independent.

The correct answer is (c)

**3.** Let $V = M_{2\times 2}(\mathbb{R})$ and $v_1 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$.

(a) Show that $v_1, v_2, v_3$ are linearly independent. (7 points)

(b) Find $v_4$ so that $S = \{v_1, v_2, v_3, v_4\}$ forms a basis of $V$. (8 points)

*Solutions:*

(a) Consider equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$. Then we have the following system of linear equations

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 2 & 3 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using elementary (row) operations to simplify the above system, we have

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So it is clear the system only has solution $x_1 = x_2 = x_3 = 0$. So $v_1, v_2, v_3$ are linearly independent.

(b) Suppose that $v_4 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and consider the equation $x_1 v_1 + x_2 v_2 + x_3 v_3 + x_3 v_4 = 0$. We get the equation

$$\begin{pmatrix} 1 & 0 & 1 & a \\ 1 & 1 & 2 & b \\ 0 & 2 & 3 & c \\ 0 & -1 & 1 & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Using elementary (row) operations to simplify the above system, we have

$$\begin{pmatrix} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 1 & c-2(b-a) \\ 0 & 0 & 0 & d-2c+5(b-a) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

So it is clear that the system only has zero solution if an only if

$$d - 2c + 5(b-a) \neq 0.$$

For example, $v_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

**4.** Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation given by $T(f(x)) = f'(x)$.

(a) Find the standard matrix representing $T$, that is, the matrix representing $T$ for the standard basis $\{1, x, x^2\}$. (5 points)

(b) Find nullity of $T$ and the rank of $T$. (5 points)

(c) Given a basis $\beta = \{1 - x, 1 + x, x^2 + x\}$. Find matrix $[T]_\beta$. (5 points)

*Solutions:*

(a) We have

$$T(1,\ x,\ x^2) = (0,\ 1,\ 2x^2) = (1,\ x,\ x^2)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

So the standard matrix of $T$ is $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) It is clear that the range $R(T)$ of $T$ is spanned by $T(1)$, $T(x)$, and $T(x^2)$. Since 1 and $2x$ are linearly independent, they forms of basis of $R(T)$ and hence $\dim_{\mathbb{R}} R(T) = \operatorname{rank}(T) = 2$. By dimension theorem, the nullity of $T$ is $\dim_{\mathbb{R}} P_2(\mathbb{R}) - \operatorname{rank}(T) = 3 - 2 = 1$.

(c) Let $\alpha = \{1, x, x^2\}$ be the standard basis and $Q = [I_{P_2(\mathbb{R})}]_\alpha^\beta$. Since $(1 - x,\ 1 + x,\ x^2 + x) = (1,\ x\ x^2)\begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ So we conclude that $Q = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

and

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix}$$

**5.** Let $A \in M_{m\times n}(F)$ be an $m \times n$-matrix. If $\text{rank}(A) = \min\{m, n\}$ then $A$ is called a *full rank matrix*. Recall that the linear transformation $L_A : F^n \to F^m$ is defined by $L_A(x) = Ax$.

(a) Suppose $m \le n$. Show that $A$ is a full rank matrix if and only rows of $A$ are linearly independent. (5 points)

(b) Suppose $m \le n$. Show that if $A$ is a full rank matrix then $L_A$ is onto. (5 points)

(c) Suppose $m \ge n$. Show that $A$ is a full rank matrix if and only if columns of $A$ are linearly independent. (5 points)

(d) Suppose $m \ge n$. Show that if $A$ is a full rank matrix then $L_A$ is one-to-one. (5 points)

*Proof:*

(a) Since $A$ is full rank, and $m \le n$, we have $\text{rank}(A) = \min\{m, n\} = m$. So $\text{rank}(A) = \dim \text{Row}(A) = m$. Since the row space $\text{Row}(A)$ is spanned by all rows of $A$ and $A$ has exactly $m$ rows, this implies that all rows of $A$ must be a basis of $\text{Row}(A)$. Hence all rows of $A$ are linearly independent. Conversely, if rows of $A$ are linearly independent, then they forms a basis of $\text{Row}(A)$. So $\text{rank}(A) = \dim \text{Row}(A) = m$, and $A$ is of full rank.

(b) By definition $\text{rank}(A) = \dim R(L_A)$. So $\dim R(L_A) = m$ as $A$ is full rank and $m \le n$. Since $R(L_A) \subset F^m$, and $\dim R(L_A) = \dim F^m = m$, we conclude that $R(L_A) = F^m$. Therefore, $L_A$ is onto.

(c) Since $A$ is full rank, and $m \ge n$, we have $\text{rank}(A) = \min\{m, n\} = n$. So $\text{rank}(A) = \dim \text{Col}(A) = n$. Since the column space $\text{Col}(A)$ is spanned by all columns of $A$ and $A$ has exactly $n$ columns, this implies that all columns of $A$ must be a basis of $\text{Col}(A)$. Hence all columns of $A$ are linearly independent. Conversely, if columns of $A$ are linearly independent, then they forms a basis of $\text{Col}(A)$. So $\text{rank}(A) = \dim \text{Col}(A) = n$, and $A$ is of full rank.

(d) If $A$ is full rank, and $m \ge n$ then we have $\text{rank}(A) = \min\{m, n\} = n$. By dimension theorem, $\dim N(L_A) + \text{rank}(L_A) = n$. So $\text{rank}(A) = \text{rank}(L_A) = n$ implies that $\dim(N(L_A)) = 0$. So $L_A$ is one-to-one.

**6.** Let $V$, $W$ be vector spaces over a field $F$ and $T : V \to W$ a linear transformation. Let $v_1, \dots, v_n \in V$ be vectors in $V$.

(a) Suppose that $T$ is one-to-one and $\{v_1, \dots, v_n\}$ are linearly independent. Show that $\{T(v_1), \dots, T(v_n)\}$ are also linearly independent. (5 points)

(b) Suppose that $T$ is onto and $\{v_1, \dots, v_n\}$ spans $V$. Show that $\{T(v_1), \dots, T(v_n)\}$ spans $W$. (5 points)

(c) Suppose that $T$ is an isomorphism and $\{v_1, \dots, v_n\}$ is a basis of $V$. Show that $\{T(v_1), \dots, T(v_n)\}$ is a basis of $W$. (5 points)

*Proof:*

(a) Consider equation of vectors $x_1 T(v_1) + \cdots + x_n T(v_n) = \vec{0}$. Since $T$ is linear, we have

$$T(x_1 v_1 + \cdots + x_n v_n) = x_1 T(v_1) + \cdots + x_n T(v_n) = \vec{0}$$

Since $T$ is one-to-one, $N(T) = \{\vec{0}\}$, we have $x_1 v_1 + \cdots + x_n v_n = \vec{0}$. But $v_1, \dots, v_n$ is linearly independent, this forces $x_1 = \cdots = x_n = 0$. So $T(v_1), \dots, T(v_n)$ is linearly independent.

(b) It suffices to show that for any $w \in W$, $w$ is linear combination of $T(v_1), \dots, T(v_n)$. Since $T$ is onto, there exists a $v \in V$ so that $w = T(v)$. Since $v_1, \dots v_n$ spans $V$, there exists linear combination $c_1 v_1 + \cdots + c_n v_n = v$ with $c_i \in F$. Then

$$w = T(v) = T(c_1 v_1 + \cdots + c_n v_n) = c_1 T(v_1) + \cdots + c_n T(v_n).$$

So $w$ is linear combination of $T(v_1), \dots, T(v_n)$ as required.

(c) Since $T$ is an isomorphism, it is one-to-one and onto. Since $\{v_1, \dots, v_n\}$ is a basis of $V$, it is linearly independent and spans $V$. By (1) and (2) above, we see that $\{T(v_1), \dots, T(v_n)\}$ is linearly independent and spans $W$. So $\{T(v_1), \dots, T(v_n)\}$ is a basis of $W$.