Math 353, Practice Midterm 2

Name: ________________________________

This exam consists of 8 pages including this front page.

Ground Rules
1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

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1. The following are true/false questions. You don’t have to justify your answers. Just write down either T or F in the table below. $A$, $B$, $C$, $X$, $b$ are always matrices here.

(a) $\det(kA) = k \det(A)$.
(b) Let $A$ be an $n \times n$-matrix. Then the linear system $AX = b$ is always consistent for all possible $b$ if and only if $A$ is invertible.
(c) If $A$ is diagonalizable then all eigenvalues of $A$ are distinct.
(d) Let $T : V \rightarrow V$ be a linear operator and $\alpha = \{v_1, \ldots, v_n\}$ an ordered basis of $V$. Then $T$ and $[T]_\alpha$ share the same eigenvalues.
(e) Let $V$ be an inner product space with inner product $\langle \cdot, \cdot \rangle$. If $\langle w, v \rangle = 0$ then either $w = 0$ or $v = 0$.

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2. Multiple Choice:

(i) Suppose that $A$ is an $m \times n$ matrix with entries in $\mathbb{R}$ and consider a system of linear equations $Ax = b$ over the field $\mathbb{R}$. Which of the following statement is NOT correct?

(a) If $\text{rank}(A) = m$ and $n > m$ then the system $Ax = b$ has infinitely many solutions.
(b) If $N(A) = \{0\}$, then $m \leq n$.
(c) If $\text{rank}(A) = n$, then $Ax = b$ has a unique solution or no solution.
(d) If $\text{rank}(A) = m$ and $n \geq m$, then $Ax = b$ has at least one solution.
(e) If $\text{rank}(A) = m$ and $n = m$, then $Ax = b$ has unique solution.

The correct answer is (b).

(ii) Which of the following is NOT equivalent to the statement that $A$ is invertible.

(a) $A$ is diagonalizable.
(b) $\det(A) \neq 0$.
(c) $A$ only has nonzero eigenvalues.
(d) $\text{rank}(A) = n$.
(e) If the characteristic polynomial $f_A(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 + a_0$ then $a_0 \neq 0$.

The correct answer is (a).

(iii) Let $V$ be a inner product space over $\mathbb{R}$. Assume that $u, v \in V$ and $\| u \| = 3$, $\| v \| = 4$. Then which of the following is correct.

(a) $u, v$ are orthogonal.
(b) $\| u - v \| \geq \min\{\| u \|, \| v \|\}$.
(c) $u + v$ and $u - v$ are orthogonal.
(d) If $\| u - v \| = 5$ then $u, v$ are orthogonal.
(e) None of the above statements.

The correct answer is (d).
(iv) Which of the following properties implies that the $n \times n$ matrix $A$ can be diagonalized?

(a) $A$ is a transition matrix.
(b) $A$ is an invertible matrix.
(c) All eigenvalues of $A$ are same.
(d) The dimension of all eigenspaces is 1.
(e) The algebraic multiplicity of eigenvalue $k_i = 1$ for all $i$.

The correct answer is (e).

(v) Consider the following linear system.

\[
\begin{align*}
  x + ay + z &= b + c \\
  2x + by + z &= a + c \\
  3x + cy + z &= a + b
\end{align*}
\]

Suppose the system only has unique solution. Then

(a) $x = 0$
(b) $y = 0$
(c) $z = 1$
(d) $x = 1$
(e) $y = 1$

The correct answer is (a).
3. Let
\[ A = \begin{pmatrix} 1 & s & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \]

(a) Find the value of \( s \) such that \( A \) is diagonalizable.

Solutions: The characteristic polynomial is
\[
P_A(\lambda) = \begin{vmatrix} \lambda - 1 & -s & 1 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)
\]
Hence the eigenvalues of \( A \) are \( \lambda_1 = \lambda_2 = 1 \) and \( \lambda_3 = 2 \). Since the eigenvalue \( \lambda_1 = 1 \) has multiplicity 2, \( A \) is diagonalizable if and only if the dimension of the 1-eigenspace \( E_1 \) is 2. Note that the \( E_1 \) is given by the solutions of \((I_3 - A)X = 0\), namely,
\[
\begin{pmatrix} 0 & -s & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]
We easily see that \( z = 0 \). So if \( s \neq 0 \) then \( y = 0 \) and then \( E_1 \) is just spanned by \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \). In this case, the dimension of \( E_1 \) is 1, which is less than the multiplicity 2. Hence \( E_1 \) has dimension 2 if and only if \( s = 0 \), in which case, \( E_1 \) has a basis \( \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \).

(b) For value \( s \) that \( A \) is diagonalizable, diagonalize \( A \). Namely, find an invertible matrix \( S \) and a diagonal matrix \( \Lambda \) such that \( A = S \Lambda S^{-1} \).

Solutions: To diagonalize \( A \), we need find eigenvectors which forms a basis. We have found the basis of \( E_1 \) from the above. It suffices to find a basis of \( E_2 \), which is the space of the solution for the following system (note \( s = 0 \) from the above question):
\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]
We easily get a basis \( \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \). So we obtain \( P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \) and
\[
\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]
4. Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation so that $T(1) = x^2 - x + 1$, $T(x) = x + 1$, $T(x^2) = -2x^2 - 4$, $T(x^3) = x^2 + 2x + 4$.

(a) Find a basis of the range $R(T)$ of $T$.
(b) Find a basis of the null space $N(T)$ of $T$.
(c) Enlarge the basis of $N(T)$ you found in the last question to a basis of $P_3(\mathbb{R})$.

Solutions: (a) Note that $R(T) = \text{Span}\{T(1), T(x), T(x^2), T(x^3)\}$. By selecting standard basis $\alpha = \{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$ and $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$, we find the matrix of $T$ is 

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & -2 & 1 \\ -1 & 1 & 0 & 2 \\ 1 & 1 & -4 & 4 \end{bmatrix}.$$ 

It is easy to compute that the reduced echelon form $R$ of $[T]_\alpha^\beta$ is

$$R = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

Since first 2 columns of $R$ have pivots, we see that column space of $[T]_\alpha^\beta$ has basis of first two columns. Since the first two columns of $[T]_\alpha^\beta$ are coordinate $[T(1)]_\beta$ and $[T(x)]_\beta$. So $T(1)$ and $T(x)$ forms a basis of $R(T)$

(b) It is not hard to see that $x \in N(T)$ if and only if $[T]_\alpha^\beta[x]_\alpha = 0$. So it suffices to find basis of $N([T]_\alpha^\beta)$. By the reduced echelon form the above, we know the $N([T]_\alpha^\beta)$ has basis

$$\begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}. \quad \text{So } 1 + 3x - x^3 \text{ and } 2 + 2x + x^2$$

forms a basis of $N(T)$.

(c) It suffices to extend the basis

$$\begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

to a basis of $\mathbb{R}^4$. So it suffices to find columns spaces of

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$
By compute echelon form of the above, we see that the first 4 columns forms a basis of column space. So $1 + 3x - x^3$, $2 + 2x + x^2$, $1$, $x$ extends to a basis of $P_3(\mathbb{R})$.

5. Show that the characteristic polynomial of

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

is

$$f_A(t) = (-1)^n(t^n + a_{n-1}t^{n-1} + \cdots + a_1t + a_0).$$

*Hint:* Use cofactor expansion along the first row and then use mathematical induction on $n$.

**Proof:** We use mathematical induction on $n$. If $n = 1$ then $f_A(t) = \begin{vmatrix} -t & -a_0 \\ 1 & -t - a_1 \end{vmatrix} = t^2 + a_1t + a_0$. Suppose $n = k$ the statement is true then for $n = k + 1$, we have

$$f_A(t) = \begin{vmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -t - a_k \end{vmatrix}.$$ 

Using cofactor expansion along the first row, we have

$$f_A(t) = -t \begin{vmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -t - a_k \end{vmatrix} + (-1)^{k+1}(-a_0).$$

By induction on $n = k$, we have

$$f_A(t) = -t ((-1)^k(t^k + a_k t^{k-1} + \cdots + a_2t + a_1)) + (-1)^{k+1}a_0$$

$$= (-1)^{k+1}(t^{k+1} + a_k t^k + \cdots + a_1t + a_0).$$

This proves the case $n = k + 1$ and hence complete the induction.
6. Let $A$ be an $n \times n$-matrix.

(a) Show that $A$ is invertible if and only if none of eigenvalues of $A$ is zero.

(b) Suppose $A$ is invertible. Show that if $\lambda$ is an eigenvalue of $A$ then $\lambda^{-1}$ is an eigenvalue of $A$.

(c) Show that $A$ is diagonalizable if and only if $A^{-1}$ is.

Proof:

(a) $A$ has an eigenvalue $\lambda = 0$ if and only if there is an eigenvector $v \neq 0$ such that $Av = \lambda v = 0$. So this is equivalent to that the homogeneous equation $AX = 0$ has nontrivial solution, which is equivalent to that $A$ is NOT invertible. So $A$ is invertible if and only if $A$ has no eigenvalue 0.

(b) For the above, we see that $\lambda \neq 0$ and $Av = \lambda v$ with $v$ being eigenvector. Timing $A^{-1}$ on the both side of $Av = \lambda v$, we have $A^{-1}Av = \lambda A^{-1}v$. That is $v = \lambda A^{-1}v$, or equivalently $A^{-1}v = \lambda^{-1}v$. So $\lambda^{-1}$ is an eigenvalue of $A^{-1}$.

(c) $A$ is diagonalizable if and only if there exists an diagonal matrix $\Lambda$ so that $A$ is similar to $\Lambda$. Or equivalently there exists an invertible matrix $S$ so that $A = SAS^{-1}$. If $A$ is invertible then all eigenvalues $\lambda_i \neq 0$. So $\Lambda$ is invertible because the diagonal of $\Lambda$ are $\lambda_i$. So we have $A^{-1} = (S\Lambda S^{-1})^{-1} = S\Lambda^{-1}S^{-1}$ with $\Lambda^{-1}$ being diagonal matrix. That is, $A^{-1}$ is also diagonalizable. Since $A = (A^{-1})^{-1}$, $A^{-1}$ is diagonalizable implies that $A$ is diagonalizable.