Math 353, Practice Midterm 2

Name: ______________________________

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	15	
2	20	
3	20	
4	15	
5	15	
6	15	
Total	100	

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. A, B, C, X, b are always matrices here.

(a) $\det(kA) = k\det(A)$.

(b) Let A be an $n \times n$-matrix. Then the linear system $AX = b$ is always consistent for *all possible* b if and only if A is invertible.

(c) If A is diagonalizable then all eigenvalues of A are distinct.

(d) Let $T : V \to V$ be an linear operator and $\alpha = \{v_1, \ldots, v_n\}$ an ordered basis of V. Then T and $[T]_\alpha$ share the same eigenvalues.

(e) Let V be an inner product space with inner product $\langle , \rangle$. If $\langle w, v\rangle = 0$ then either $w = 0$ or $v = 0$.

	(a)	(b)	(c)	(d)	(e)
Answer	F	T	F	T	F

2. Multiple Choice:

(i) Suppose that A is an $m \times n$ matrix with entries in $\mathbb{R}$ and consider a system of linear equations $Ax = b$ over the field $\mathbb{R}$. Which of the following statement is NOT correct?

(a) If $\text{rank}(A) = m$ and $n > m$ then the system $Ax = b$ has infinitely many solutions.
(b) If $N(A) = \{0\}$, then $m \leq n$.
(c) If $\text{rank}(A) = n$, then $Ax = b$ has a unique solution or no solution.
(d) If $\text{rank}(A) = m$ and $n \geq m$, then $Ax = b$ has at least one solution.
(e) If $\text{rank}(A) = m$ and $n = m$, then $Ax = b$ has unique solution.

The correct answer is (b)

(ii) Which of the following is NOT equivalent to the statement that A is invertible.

(a) A is diagonalizable.
(b) $\det(A) \neq 0$.
(c) A only has nonzero eigenvalues.
(d) $\text{rank}(A) = n$.
(e) If the characteristic polynomial $f_A(t) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_t + a_0$ then $a_0 \neq 0$.

The correct answer is (a).

(iii) Let V be a inner product space over $\mathbb{R}$. Assume that $u, v \in V$ and $\| u \| = 3$, $\| v \| = 4$. Then which of the following is correct.

(a) u, v are orthogonal.
(b) $\| u - v \| \geq \min\{\| u \|, \| v \|\}$.
(c) $u + v$ and $u - v$ are orthogonal.
(d) If $\| u - v \| = 5$ then u, v are orthogonal.
(e) None of the above statements.

The correct answer is (d).

(iv) Which of the following properties implies that the $n \times n$ matrix A can be diagonalized?

(a) A is a transition matrix.
(b) A is an invertible matrix.
(c) All eigenvalues of A are same.
(d) The dimension of all eigenspaces is 1.
(e) The algebraic multiplicity of eigenvalue $k_i = 1$ for all i.

The correct answer is (e).

(v) Consider the following linear system.

$$x + ay + z = b + c$$

$$2x + by + z = a + c$$

$$3x + cy + z = a + b$$

Suppose the system only has unique solution. Then

(a) $x = 0$
(b) $y = 0$
(c) $z = 1$
(d) $x = 1$
(e) $y = 1$

The correct answer is (a).

3. Let

$$A = \begin{pmatrix} 1 & s & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

(a) Find the value of s such that A is diagonalizable.

Solutions: The characteristic polynomial is

$$P_A(\lambda) = \begin{vmatrix} \lambda - 1 & -s & 1 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)$$

Hence the eigenvalues of A are $\lambda_1 = \lambda_2 = 1$ and $\lambda_3 = 2$. Since the eigenvalue $\lambda_1 = 1$ has multiplicity 2, A is diagonalizable if and only if the dimension of the 1-eigenspace E_1 is 2. Note that the E_1 is given by the solutions of $(1I_3 - A)X = 0$, namely,

$$\begin{pmatrix} 0 & -s & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We easily see that $z = 0$. So if $s \neq 0$ then $y = 0$ and then E_1 is just spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. In this case, the dimension of E_1 is 1, which is less than the multiplicity 2. Hence E_1 has dimension 2 if and only if $s = 0$, in which case, E_1 has a basis $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(b) For value s that A is diagonalizable, diagonalize A. Namely, find an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.

Solutions: To diagonalize A, we need find eigenvectors which forms a basis. We have found the basis of E_1 from the above. It suffices to find a basis of E_2, which is the space of the solution for the following system (note $s = 0$ from the above question):

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We easily get a basis $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. So we obtain $P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $\Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

4. Let $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ be a linear transformation so that $T(1) = x^2 - x + 1$, $T(x) = x + 1$, $T(x^2) = -2x^2 - 4$, $T(x^3) = x^2 + 2x + 4$.

(a) Find a basis of the range $R(T)$ of T.

(b) Find a basis of the null space $N(T)$ of T.

(c) Enlarge the basis of $N(T)$ you found in the last question to a basis of $P_3(\mathbb{R})$.

Solutions: (a) Note that $R(T) = \text{Span}\{T(1), T(x), T(x^2), T(x^3)\}$. By selecting standard basis $\alpha = \{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$ and $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$, we find the matrix of T is

$$[T]_\alpha^\beta = \begin{bmatrix} 1 & 0 & -2 & 1 \\ -1 & 1 & 0 & 2 \\ 1 & 1 & -4 & 4 \end{bmatrix}.$$

It is easy to compute that the reduced echelon form R of $[T]_\alpha^\beta$ is

$$R = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since first 2 columns of R have pivots, we see that column space of $[T]_\alpha^\beta$ has basis of first two columns. Since the first two columns of $[T]_\alpha^\beta$ are coordinate $[T(1)]_\beta$ and $[T(x)]_\beta$. So $T(1)$ and $T(x)$ forms a basis of $R(T)$

(b) It is not hard to see that $x \in N(T)$ if and only if $[T]_\alpha^\beta [x]_\alpha = 0$. So it suffices to find basis of $N([T]_\alpha^\beta)$. By the reduced echelon form the above, we know the $N([T]_\alpha^\beta)$ has basis $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$. So $1 + 3x - x^3$ and $2 + 2x + x^2$ forms a basis of $N(T)$.

(c) It suffices to extend the basis $\begin{bmatrix} 1 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$ to a basis of $\mathbb{R}^4$. So it suffices to find columns spaces of

$$\begin{bmatrix} 1 & 2 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

By compute echelon form of the above, we see that the first 4 columns forms a basis of column space. So $1+3x-x^3$, $2+2x+x^2$, 1 , x extends to a basis of $P_3(\mathbb{R})$.

5. Show that the characteristic polynomial of

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$$

is

$$f_A(t) = (-1)^n(t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0).$$

Hint: Use cofactor expansion along the first row and then use mathematical induction on n.

Proof: We use mathematical induction on n. If $n=1$ then $f_A(t) = \begin{vmatrix} -t & -a_0 \\ 1 & -t-a_1 \end{vmatrix} = t^2 + a_1 t + a_0$. Suppose $n = k$ the statement is true then for $n = k+1$, we have

$$f_A(t) = \begin{vmatrix} -t & 0 & \cdots & 0 & -a_0 \\ 1 & -t & \cdots & 0 & -a_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -t-a_k \end{vmatrix}.$$

Using cofactor expansion along the first row, we have

$$f_A(t) = -t\begin{vmatrix} -t & 0 & \cdots & 0 & -a_1 \\ 1 & -t & \cdots & 0 & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -t-a_k \end{vmatrix} + (-1)^{k+2}(-a_0)\begin{vmatrix} 1 & -t & \cdots & 0 & -a_1 \\ 0 & 1 & -t & \cdots & -a_2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & & 1 \end{vmatrix}.$$

By induction on $n = k$, we have

$$\begin{aligned} f_A(t) &= -t\left((-1)^k(t^k + a_k t^{k-1} + \cdots + a_2 t + a_1)\right) + (-1)^{k+1}a_0 \\ &= (-1)^{k+1}(t^{k+1} + a_k t^k + \cdots + a_1 t + a_0). \end{aligned}$$

This proves the case $n = k+1$ and hence complete the induction.

6. Let A be an $n \times n$-matrix.

(a) Show that A is invertible if and only if none of eigenvalues of A is zero.

(b) Suppose A is invertible. Show that if λ is an eigenvalue of A then λ^{-1} is an eigenvalue of A.

(c) Show that A is diagonalizable if and only if A^{-1} is.

Proof:

(a) A has an eigenvalue $\lambda = 0$ if and only if there is an eigenvector $v \neq 0$ such that $Av = \lambda v = 0$. So this is equivalent to that the homogeneous equation $AX = 0$ has nontrivial solution, which is equivalent to that A is NOT invertible. So A is invertible if and only if A has no eigenvalue 0.

(b) For the above, we see that $\lambda \neq 0$ and $Av = \lambda v$ with v being eigenvector. Timing A^{-1} on the both side of $Av = \lambda v$, we have $A^{-1}Av = \lambda A^{-1}v$. That is $v = \lambda A^{-1}v$, or equivalently $A^{-1}v = \lambda^{-1}v$. So λ^{-1} is an eigenvalue of A^{-1}.

(c) A is diagonalizable if and only if there exists an diagonal matrix Λ so that A is similar to Λ. Or equivalently there exists an invertible matrix S so that $A = S\Lambda S^{-1}$. If A is invertible then all eigenvalues $\lambda_i \neq 0$. So Λ is invertible because the diagonal of Λ are λ_i. So we have $A^{-1} = (S\Lambda S^{-1})^{-1} = S\Lambda^{-1}S^{-1}$ with Λ^{-1} being diagonal matrix. That is, A^{-1} is also diagonalizable. Since $A = (A^{-1})^{-1}$, A^{-1} is diagonalizable implies that A is diagonalizable.