Math 353, Midterm 2

Name: ______________________________

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.
3. You may use one 3-by-5 index card, both sides.

Score		
1	15	
2	20	
3	20	
4	15	
5	15	
6	15	
Total	100	

Notations: $\mathbb{R}$ denotes the set of real number and $\mathbb{C}$ denotes the set of complex numbers; F is always a field, for example, $F = \mathbb{R}$; $M_{m\times n}(F)$ denotes the set of $m \times n$-matrices with entries in F; $F^n = M_{n\times 1}(F)$ denotes the set of n-column vectors; $P_n(F)$ denotes the set of polynomials with coefficients in F and the most degree n, that is,

$$P_n(F) = \{f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_i \in F, \ \forall i\}.$$

V is always a finite dimensional vector space over F and T is always a linear operator $T : V \to V$.

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. (3 points each)

(a) If $\det(A) = 0$ then columns of A are linearly dependent.

(b) Let $v_1, \ldots, v_m$ be eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_m$. Suppose that $\lambda_1, \ldots, \lambda_m$ are distinct then $v_1, \ldots, v_m$ are linearly independent.

(c) Let V be a inner product space and $y, z \in V$. If $\langle x, y\rangle = \langle x, z\rangle$ for all $x \in V$ then $y = z$.

(d) If a given linear system has 5 unknowns x_i and 7 equations, then the system must be inconsistent.

(e) A square matrix A is invertible if and only if 0 is not an eigenvalue of A.

	(a)	(b)	(c)	(d)	(e)
Answer	T	T	T	F	T

2. Multiple Choice. (4 points each)

(i) Consider the following linear system.

$$\begin{aligned} x + ay + z &= a + 1 \\ 2x + by + z &= b + 1 \\ 3x + cy + z &= c + 1 \end{aligned}$$

Suppose the system only has unique solution. Then

(a) $x = 1$
(b) $y = 0$
(c) $z = 1$
(d) $x = 2$
(e) $y = 2$

The correct answer is (c).

(ii) Let

$$A = \begin{pmatrix} 0 & 7 & a & 1 \\ 0 & 2 & 0 & 0 \\ 3 & 4 & 5 & 6 \\ 0 & 8 & 9 & a \end{pmatrix}$$

Which of the following statement is correct?

(a) $\det(A) = -6(a^2 - 9)$
(b) $\det(A) = 6(a^2 - 9)$
(c) $\det(A) = 0$.
(d) A is always invertible.
(e) A is invertible if and only if $a \neq 3$.

The correct answer is (a).

(iii) Let $C[-1,1]$ be the space of all real continuous functions over $[-1,1]$ with inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(t)g(t)dt.$$

Which of the following set is orthonormal?

(a) $1, t, t^2$.

(b) $\sin t, \cos t$.

(c) $1, e^t$.

(d) $\frac{1}{2}, \frac{t}{\sqrt{2/3}}$.

(e) None of the above.

The correct answer is (e).

(iv) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Which of the following statement is correct?

(a) A is invertible.

(b) Eigenvalues of A are all distinct.

(c) A is NOT diagonalizable.

(d) All the eigenspaces of A have the same dimension.

(e) $A^3 - 3A^2 = 0$.

The correct answer is (e).

(v) Suppose that A is an $m \times n$ matrix with entries in $\mathbb{R}$ and consider a system of linear equations $Ax = b$ over the field $\mathbb{R}$. Which of the following statement is correct?

(a) If $\text{rank}(A) = m$ and $n > m$ then the system $Ax = b$ has a unique solution.

(b) If $N(A) = \{0\}$ then $m \leq n$.

(c) If $\text{rank}(A) = n$ then $Ax = b$ must has a unique solution.

(d) If $\text{rank}(A) = m$ and $n \geq m$, then $Ax = b$ has at least one solution.

(e) If $\text{rank}(A) = m$ and $n = m$, then $Ax = b$ could have no solution.

The correct answer is (d).

3. Let $\mathbb{R}_4 := \{(a_1, a_2, a_3, a_4) | a_i \in \mathbb{R}\}$ be the space of 4-row vectors. Consider

$$S = \{v_1 = (1\ 0\ 2\ 1), v_2 = (-1\ 1\ 0\ 2), v_3 = (1\ 1\ 4\ 4)\})$$

(a) Find a basis of Span$S$ (5 points)

(b) Extend the basis of Span$S$ found in (a) to a basis of $\mathbb{R}_4$. (5 points)

(c) Consider the standard inner product on $\mathbb{R}_4$ (i.e., if $x = (a_1, a_2, a_3, a_4)$ and $y = (b_1, b_2, b_3, b_4)$ then $\langle x, y \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4$) and define

$$W = \{x \in \mathbb{R}_4 | x, v_i \text{ are orthogonal for all } i = 1, 2, 3\}.$$

(i) Show that W is a subspace of $\mathbb{R}_4$. (5 points)

(ii) Find a basis of W. (5 points)

Solutions: (a) Consider $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ -1 & 1 & 0 & 2 \\ 1 & 1 & 4 & 4 \end{bmatrix}$. To find basis of Span$S$, it is equivalent to find row space of A. For this, we find the reduce echelon form of A: $R = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since row elementary operation does not change row space, the first two rows of R forms a basis of Span$S$.

Alternate method: Consider equation of vectors: $x_1 v_1 + x_2 v_2 + x_3 v_3 = \vec{0}$ which is equivalent to $A^T X = 0$. Find reduced echelon form of A^T, which has pivots in the first 2 columns. Then v_1, v_2 forms a basis of Span$S$.

(b) Let w_1, w_2 denote the basis found in step (a). It suffices to find basis of Span$\{w_1, w_2, e_1, e_2, e_3, e_4\}$. This is equivalent to find row space of

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

By using row echelon form of the above, it is hard to see that w_1, w_2, e_1, e_2 is a basis required.

(c) (i)& (ii): Now that $\langle x, v_i \rangle = v_i x^T$. So x, v_i are orthogonal is equivalent to that $v_i x^T = 0$ for all v_i. So W is isomorphic (via $x \mapsto x^T$) to

$$\{X \in \mathbb{R}^4 | AX = \vec{0}\} = N(A).$$

So W is a subspace. To find a basis of W, it is equivalent to find a basis of $N(A)$, which can be read from echelon form of A. Then it is not hard to see that $(-2\ \ -2\ 1\ 0), (-1\ \ -3\ 0\ 1)$ forms a basis of W.

4. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear operator given by

$$T(f(x)) = f'(x) + 2f(x).$$

(a) Find all eigenvalues λ_i of T . (5 points)

(b) For each eigenvalue λ_i, find a basis of eigenspace

$$E_{\lambda_i} = \{v \in P_2(\mathbb{R}) | T(v) = \lambda_i v\}. \text{ (5 points)}$$

(c) Is T diagonalizable? Why or why not? (5 points)

Solutions: a) Take the standard basis $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$, the matrix $A = [T]_\beta$ representing the operator T is determined by

$$T(1, x, x^2) = (0, 1, 2x) = (1, x, x^2) \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

It suffices to find eigenvalues of $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$.

We easily see the characteristic polynomial of A is $f_A(t) = (2-t)^3$. So eigenvalues of A is 2 with algebraic multiplicity 3.

b) Now $\lambda_i = 2$, to find the eigenspace

$$E_2 = \{v \in P_2(\mathbb{R}) | T(v) = \lambda_i v = 2v\}.$$

We first find the eigenspace E'_2 of A for eigenvalue 2. By solving $(A-2I)X = \vec{0}$, we easily find that E'_2 has dimension 1 and spanned by It is clear that $T(f(x)) = 0$ if and only if $f(x) = c$ with c a constant in $\mathbb{R}$. So $f(x) = 1$ is a basis of the eigenspace E_0. One can also find eigenvector w of A, and it is easy to see that $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. So E_2 also has dimension 1 and spanned by $f(x) = 1 + 0x + 0x^2 = 1$.

c) Since algebraic multiplicity of $\lambda = 2$ is 3 is larger than the geometric multiplicity $\dim_F E_2 = 1$. So T is NOT diagonalizable.

5. Let A be an $n \times n$-matrix.

(a) Show that if v is an eigenvector of A with eigenvalue λ then v is also an eigenvector of A^m with eigenvalue λ^m. (5 points)

(b) Let $f(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ be a polynomial and I_n be the identity matrix. Define

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_n.$$

Show that if λ is an eigenvalue of A then $f(\lambda)$ is an eigenvalues of $f(A)$. (5 points)

(c) Show that if A is diagonalizable then A^m is also diagonalizable.(5 points)

Proof: (a) By the definition of eigenvalue and eigenvector, $Av = \lambda v$. So

$$A^m v = A^{m-1}(Av) = A^{m-1}\lambda v = \lambda A^{m-1} v = \lambda A^{m-2} A v = \lambda A^{m-2} \lambda v = \cdots = \lambda^m v$$

Since $v \neq \vec{0}$, v is an eigenvector of A^m with eigenvalue λ^m.

(b) Since $A^m v = \lambda^m v$ as the above, we see that

$$\begin{aligned} f(A)v &= (a_n A^n + a_{n-1} A^{n-1} + \cdots + a_1 A + a_0 I_n) v \\ &= a_n A^n v + a_{n-1} A^{n-1} v + \cdots + a_1 A v + a_0 I_n v \\ &= a_n \lambda^n v + a_{n-1} \lambda^{n-1} v + \cdots + a_1 \lambda v + a_0 v \\ &= (a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0) v \\ &= f(\lambda) v. \end{aligned}$$

So v is an eigenvector of $f(A)$ with eigenvalue $f(\lambda)$.

(c) A is diagnolizable if and only if there exists an invertible matrix S so that $A = S\Lambda S^{-1}$ with Λ a diagonal matrix. Now

$$A^m = S\Lambda S^{-1} S\Lambda S^{-1} \cdots S\Lambda S^{-1} = S\Lambda^m S^{-1}.$$

Since Λ^m is a diagonal matrix, A^m is diagonalizable.

6. Let A be an $n \times n$-matrix and $\lambda_1, \dots, \lambda_n$ all its eigenvalues (λ_i may not be distinct). Let us show that

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

(a) Show the above statement is true if A is diagonalizable. (5 points)

(b) The proof for the general A is more challenging with following steps:

(i) Let $f_A(t) = (-1)^n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$ be the characteristic polynomial of A. Show that $a_0 = f_A(0) = \det(A)$ (5 points).

(ii) Using that λ_i are roots of $f_A(t)$ to prove $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$. (5 points).