Math 453, Practice Midterm 1

Name: ____

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	15	
2	16	
3	20	
4	18	
5	18	
6	13	
Total	100	

Notations: In the following, $\mathbb{Z}$ denotes the group set of integers with addition, $\mathbb{Z}_n := \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$ denote the group of $\mathbb{Z}$ modulo n, S_n denotes the group of all n-permutation. $\mathrm{GL}_n(R)$ denotes the group of $n \times n$-matrices with entries in the set $\mathbb{R}$ of real numbers. G, H, K are always groups unless otherwise stated. Let $f : G \to H$ be a homomorphism. Then $\ker(F) := \{x \in G | f(x) = e\}$ denotes the kernel of f and $f(G) := \{y | y = f(x) \text{ for some } x \in G\}$ denotes the range of G.

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. (3 points each).

(a) Let H_1, H_2 be subgroup of a group G , then $H_1 \cap H_2$ is a subgroup of G.

(b) Let $a, x, y \in G$ then $ax = ya$ implies that $x = y$.

(c) Any subgroup in an abelian group G is a normal subgroup.

(d) If a permutation $\pi \in S_n$ can be written $\pi = t_1 \ldots t_m$ with $2|m$ and t_i are *cycles* then π is even.

(e) Let $f : G \to H$ be an *isomorphism*. If G is abelian, so is H.

	(a)	(b)	(c)	(d)	(e)
Answer	T	F	T	F	T

2. Multiple Choice, (4 points each):

(i) Let $G = \langle a \rangle$ be a finite group. Which of the following statement is NOT correct.

(a) $|G| = \text{ord}(a)$.
(b) Any subgroup of G is cyclic.
(c) a is a generator of G.
(d) a is the unique generator of G.
(e) If $n = \text{ord}(a)$ then $G = \{a^i | 0 \leq i < n\}$.

The correct answer is (d).

(ii) Which of the given subsets of $\text{GL}_2(\mathbb{R})$ is a NOT subgroup?

(a) The set of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.
(b) The set of all symmetric matrices, that is $A^T = A$.
(c) The set of all matrices so that $\det(A) = 1$.
(d) The set of all matrices of the form $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.
(e) The set of all all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$.

The correct answer is (b).

(iii) Which of the following relation $\sim$ is an equivalence relation on $\mathbb{R}$.

(a) $x \sim y$ if $x \geq y$
(b) $x \sim y$ if $x = y^2$
(c) $x \sim y$ if $x + y < 0$.
(d) $x \sim y$ if $x - y \in \mathbb{Z}$.
(e) $x \sim y$ if $\max x, y = x$.

The correct answer is (d).

(iv) Let $f : G \to H$ be a homomorphism. Which of the following statement is always correct.

(a) If G is abelian so is $f(G)$.

(b) If H is finite so is $\ker f$.
(c) If H is cyclic so is $\ker f$.
(d) If a has order n so is $f(a)$.
(e) $f(G)$ is a normal subgroup of H.

The correct answer is (a).

3. Let $\pi := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix} \in S_6$ (5 points each).

(a) write π as a product of disjoint cycles.

(b) Find $\mathrm{ord}(\pi)$.

(c) Is π even or odd?

(d) Find all permutations inside $G = \langle \pi \rangle$.

Solutions:

a) $\pi = (153)(264)$.

b) $\mathrm{ord}(\pi) = 3$

c) π is even because each cycle of length 3 is even.

d) Since π has order 3, we have $G = \{e, \pi, \pi^2\}$ where $\pi^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 1 & 2 \end{pmatrix}$.

4. Let G and H be groups (6 points each).

(a) Consider $G \times H : \{(a, x) | a \in G, x \in H\}$. Show that $G \times H$ together with operation $(a_1, x_1)(a_2, x_2) = (a_1a_2, x_1x_2)$ is a group.

(b) Let $f : G \to H$ be a homomrohism. Show that the function $\tilde{f} : G \to G \times H$ defined by $\tilde{f}(a) = (a, f(a))$ is a homomorphism.

(c) Show that $\tilde{f}$ is injective.

Proof: a) It is clear that the rule $(a_1, x_1)(a_2, x_2) = (a_1a_2, x_1x_2)$ on $G \times H$ is an operation because the multiplications on G and H are operations. To show that $G \times H$ is a group: We need to check

(i) associativity: $((a_1, x_1)(a_2, x_2))(a_3, x_3) = (a_1, x_1)((a_2, x_2)(a_3, x_3))$.

(ii) Identity: $\exists e \in G \times H$ so that $e(a, x) = (a, x)e = (a, x)$;

(iii) Inverse: $\forall (a, x) \in G \times H$, $\exists (b, y)$ so that $(b, y)(a, x) = (a, x) = e$.

For (i), we have

$$\begin{aligned} ((a_1, x_1)(a_2, x_2))(a_3, x_3) = (a_1a_2, x_1x_2)(a_3, x_3) &= (a_1a_2a_3, x_1x_2x_3) = (a_1, x_1)(a_2a_3, x_2x_3) \\ &= (a_1, x_1)((a_2, x_2)(a_3, x_3)). \end{aligned}$$

For (ii), let e_1 and e_2 be identities of G and H respectively and set $e = (e_1, e_2)$. Then

$$e(a, x) = (e_1a, e_2x) = (a, x) = (ae_1, xe_2) = (a, x)e.$$

For (iii), let $b = a^{-1}$ and $y = x^{-1}$. Then

$$(a, x)(b, y) = (ab, xy) = (e_1, e_2) = e = (ba, yx) = (b, y)(a, x).$$

So $G \times H$ with the given operation is a group.

b) Since $f : G \to H$ is a function, we have that $\tilde{f} : G \to G \times H$ given by $f(a) = (a, f(a))$ is a function. Since f is homomorphism, $f(ab) = f(a)f(b)$. So

$$\tilde{f}(ab) = (ab, f(ab)) = (abf(a), f(b)) = (a, f(a))(b, f(b)) = \tilde{f}(a)\tilde{f}(b).$$

So $\tilde{f}$ is a homomorphism.

c) Consider $\ker(\tilde{f}) = \{(a) | \tilde{f}(a) = e = (e_1, e_2)\}$. Note that $\tilde{f}(a) = e$ implies that $(a, f(a)) = (e_1, e_2)$. Hence $a = e_1$, that is $\ker(\tilde{f}) = \{e_1\}$. So $\tilde{f}$ is injective.

5. Let $G = \langle a \rangle$ be a cyclic group with $|G| = n$ (6 points each).

(a) For a $k \in \mathbb{Z}$, and $l = \gcd(k, n)$ and $m = n/l$. Show that $\text{ord}(a^k) = m$.

(b) Show that a^k is generator of G if and only if k and n is relatively prime.

(c) Show that if n is a prime then G contains only trivial subgroups $\{e\}$ and G.

proof: a) Write $k = lk'$ and $n = lm$. Then $mk = mlk' = nk'$. So

$$(a^k)^m = a^{km} = a^{nk'} = (a^n)^{k'} = 1.$$

So $\text{ord}(a^k)|m$.

On the other hand, suppose $s = \text{ord}(a^k)$. Then $(a^k)^s = a^{ks} = 1$. So $n|ks$. Then $lm|lk's$ and hence $m|k's$. Since $l = \gcd(k, n)$, k' and m are relatively prime. This implies that $m|s$. Combining the fact that $s|m$, we have $s = \text{ord}(a^k) = m$.

b) a^k generates G if and only if $\text{ord}(a^k) = n$ which means $m = n/\gcd(k, n) = n$. But this is equivalent that $\gcd(k, n) = 1$, that is k and n are relatively prime.

c) Since n is prime, for $0 \leq n < n$, then k and n are always relatively prime unless $k = 0$. Since all subgroup H of G is cyclic and $H = \langle a^k \rangle$ for some k. Then either $k \neq 0$ so that a^k is a generator of G and then $H = \langle a^k \rangle = G$, or $k = 0$ and $H = \langle e \rangle = \{e\}$.

6. Let H and K be subgroups of G. Set $HK := \{ax | a \in H,\ x \in K\}$

(a) Suppose that K is a normal subgroup of G then HK is a subgroup of G (8 points).

(b) Can we drop the assumption that K is normal? Prove or disprove your statement (5 points).

Proof: a) We must check HK is closed under multiplication and inverse. For any ax, $by \in HK$, with $a, b \in H$ and $x, y \in K$, we have

$$axby = ab(b^{-1}xb)y.$$

Noe that K is a normal subgroup of G, then $(b^{-1}xb) \in K$ and hence $(b^{-1}xb)y \in K$. So $ab(b^{-1}xb)y \in HK$ because $ab \in H$. Thus HK is closed under multiplication.

Now $(ax)^{-1} = x^{-1}a^{-1} = a^{-1}(ax^{-1}a^{-1})$. Since $ax^{-1}a^{-1} = (a^{-1})^{-1}x^{-1}a^{-1}$ and $x^{-1} \in K$, by that K is normal, we have $ax^{-1}a^{-1} \in K$. So $(ax)^{-1} = x^{-1}a^{-1} = a^{-1}(ax^{-1}a^{-1}) \in HK$. This implies that HK is closed under inverse and then HK is a subgroup of G.

b) We can not drop the assumption that K is normal subgroup of G. For example, let $G = \mathrm{GL}_2(\mathbb{R})$. $H = \{\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} | x \in \mathbb{R}\}$ and $K := \{I_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$. Here K is an cyclic group with order 2. Notice that

$$HK = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix} |x, y \in R \right\}$$

This is not a subgroup of G because $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 1 \\ 1 & 0 \end{pmatrix}$ is not inside HK. In particular, K can not be normal.