Math 453, Midterm 1

Name: 

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

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Notations: In the following, $\mathbb{Z}$ denotes the group set of integers with addition, $\mathbb{Z}_n := \{0, 1, \ldots, n-1\}$ denote the group of $\mathbb{Z}$ modulo $n$, $S_n$ denotes the group of all $n$-permutations. $\text{GL}_n(R)$ denotes the group of invertible $n \times n$-matrices with entries in the set $\mathbb{R}$ of real numbers. $G, H, K$ are always groups unless otherwise stated. Let $f : G \to H$ be a homomorphism. Then $\ker(f) := \{x \in G | f(x) = e\}$ denotes the kernel of $f$ and $f(G) := \{y | y = f(x) \text{ for some } x \in G\}$ denotes the range of $G$.

1. The following are true/false questions. You don’t have to justify your answers. Just write down either T or F in the table below. (3 points each).

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2. Multiple Choice, (4 points each):

(i) Which of the following relation \(\sim\) is an equivalence relation on \(\mathbb{R}^* = \{x \in \mathbb{R}, x \neq 0\}\).

(a) \(x \sim y\) if \(x \geq y\)
(b) \(x \sim y\) if \(x = y^2\)
(c) \(x \sim y\) if \(xy < 0\).
(d) \(x \sim y\) if \(x/y\) is a rational number.
(e) \(x \sim y\) if \(\min\{x, y\} = x\).

The correct answer is D

(ii) Which of the following groups are isomorphic?

(a) \(\mathbb{Z}_4\) and the group consisting of rotations of a square which keeps the square’s position.
(b) \(\mathbb{Z}_n\) and \(\mathbb{R}^* = \{x \in \mathbb{R}, x \neq 0\}\) with multiplication.
(c) \(\text{GL}_n(\mathbb{R})\) and \(\mathbb{Z}\).
(d) \(S_n\) and a cyclic group \(G\) with \(|G| = n\).
(e) \(S_3\) and the group consisting rotations of a square which keeps the square’s position.

The correct answer is A.
(iii) Let $\pi := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix} \in S_6$. Which of the following statement is NOT correct?

(a) $\pi$ is a product of two disjoint cycles.
(b) $\pi$ is even.
(c) The order of $\pi$ is 3.
(d) The subgroup generated by $\pi$ has 4 elements.
(e) $\pi^2$ is a generator of $\langle \pi \rangle$.

The correct answer is D.

(iv) Let $G = \{e, a, b, b^2, ab, ab^2\}$ be a group whose generators $a$ and $b$ satisfy the equations,

$$a^2 = e, \quad b^3 = e, \quad ba = ab^2.$$  

Which of the following statement is NOT correct:

(a) $G$ can not be isomorphic to $\mathbb{Z}_n$ for some $n$.
(b) $G$ is not abelian.
(c) $G$ is isomorphic to a subgroup of $S_6$.
(d) $\text{ord}(ab) = 3$.
(e) $\text{ord}(ab^2) = 2$.

The correct answer is D.
3. Let $G = \mathbb{Z}_{12} := \{\bar{0}, \bar{1}, \bar{2}, \cdots, \bar{11}\}$. (5 points each).

(a) Find the orders of $\bar{3}$ and $\bar{8}$.
(b) Find all $i$ such that $\mathbb{Z}_{12} = \langle \bar{i} \rangle$.
(c) Find all subgroups of $\mathbb{Z}_{12}$.
(d) Let $\mathbb{R}^* := \{x \in \mathbb{R} | x \neq 0\}$ with multiplication. Can $\mathbb{Z}_{12}$ be isomorphic to a subgroup of $\mathbb{R}^*$. Why or why not?

Solutions

(a) $\text{ord}(\bar{3}) = 4$, $\text{ord}(\bar{8}) = 3$.
(b) Recall that $a^k$ is generator of $\langle a \rangle$ with order $n$ if and only if $k$ and $n$ are relatively prime. So $\bar{1}, \bar{5}, \bar{7}, \bar{11}$ has the property that $\mathbb{Z}_{12} = \langle \bar{i} \rangle$.
(c) Note that subgroups of a cyclic group is still cyclic, we have the following subgroups: $\{0\}, \mathbb{Z}_{12}, \langle \bar{2} \rangle, \langle \bar{3} \rangle, \langle \bar{4} \rangle, \langle \bar{6} \rangle$.
(d) No, otherwise if $\mathbb{R}^*$ has a subgroup $H$ which is isomorphic to $\mathbb{Z}_{12}$. Then $\mathbb{R}^*$ contains an elements $x$ with order $12$. This means $x^{12} = 1$. That is $x = e^{2\pi i/n}$ which is not inside $\mathbb{R}$. 
4. Let $G$ be a group (6 points each).

(a) Show that $f : G \to G$ via $f(x) = x^{-1}$ is an isomorphism if and only if $G$ is abelian.

(b) Let $f_a : G \to G$ be a function given by $f(x) = x^a$ with $a \in \mathbb{Z}$. Show that if $G$ is abelian then $f_a$ is a homomorphism.

(c) Assume that $G$ is abelian. When $f_a$ is injective? prove your statement.

\textit{proof}

(a) If $G$ is abelian then

$$f(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = f(x)f(y).$$

So $f : G \to G$ is a homomorphism. To check that $f$ is bijective, this is equivalent to check that for any $y \in G$, there exists a unique $x$ so that $f(x) = y$. But $f(y^{-1}) = (y^{-1})^{-1} = y$ and the inverse of $y$ is always unique. So $f$ is bijective and is an isomorphism.

Conversely, if $f$ is an isomorphism. Then for any $x, y \in G$, we have

$$f(xy) = (xy)^{-1} = y^{-1}x^{-1} = f(x)f(y) = x^{-1}y^{-1}. $$

Note that $y^{-1}x^{-1} = x^{-1}y^{-1}$ is equivalent to that $xy = yx$. So $G$ is abelian.

(b) Since $G$ is abelian, we have $(xy)^a = x^a y^a$. Therefore, $f_a(xy) = (xy)^a = x^a y^a = f_a(x)f_a(y)$ and thus $f_a$ is a homomorphism.

(c) $f_a$ is injective if and only if $\ker(f_a) = \{e\}$. Since

$$\ker(f_a) = \{x|x^a = e\} = \{x \in G|\text{ord}(x)|a\},$$

$f_a$ is injective if and only if when $\text{ord}(x)|a$ then $x = e.$
5. Let $G$ be a group, $a \in G$. Set $C_a := \{ x \in G | ax = xa \}$. (6 points each)

(a) Show that $C_a$ is a subgroup of $G$.

(b) Let $G = \text{GL}_2(\mathbb{R})$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Find $C_a$.

(c) Is $C_a$ always a normal subgroup? Prove or disprove your statement.

Proof

(a) We need to check that $C_a$ is closed under multiplication and inverse. For any $x, y \in C_a$, then

$$(xy)a = x(ya) = xay = (xa)y = axy.$$  

and hence $xy \in C_a$. This prove that $C_a$ is closed under multiplication. Since $xa = ax$, we have $xax^{-1} = axx^{-1} = a$, and then $x^{-1}xax^{-1} = x^{-1}a$. So $ax^{-1} = x^{-1}a$ and $x^{-1} \in C_a$. Thus $C_a$ is closed under inverse and $C_a \subset G$ is a subgroup.

(b) Here $C_a = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$. This forces that $b = c = 0$. So $C_a = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} | a, d \in \mathbb{R} \right\}$.

(c) $C_a$ is not always normal subgroup of $G$. In fact, in the above example $C_a = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} | a, d \in \mathbb{R} \right\}$ is not a normal subgroup of $\text{GL}_2$. Indeed let $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$, $y = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in C_a$, we have

$$xyx^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \notin C_a.$$
6. Let \( G, H \) be groups. Recall that \( G \times H = \{(a, x) | a \in G, x \in H\} \) together with operation \((a, x)(b, y) = (ab, xy)\) is a group. Now let \( G = \langle a \rangle \) and \( H = \langle b \rangle \) be cyclic groups with \( \text{ord}(a) = m \) and \( \text{ord}(b) = n \). Consider \( G \times H = \langle a \rangle \times \langle b \rangle \).

(a) Show that \( G \times H \) is abelian and \( |G \times H| = mn \). (5 points)

(b) Let \( l = \gcd(m, n) \). Show that \( \text{ord}((a, b)) = \frac{mn}{l} \). (4 points)

(c) Show that if \( m, n \) are relatively prime then \( G \times H \) is cyclic. (4 points)

\[ \text{proof:} \]

(a) Since \( G \) and \( H \) are cyclic, \( G \) and \( H \) are abelian. Then for \((a, x), (b, y) \in G \times H\),

\[ (a, x)(b, y) = (ab, xy) = (ba, yx) = (b, y)(a, x), \]

Thus \( G \times H \) is abelian. Note that \( |G \times H| = |G||H| = \text{ord}(a)\text{ord}(b) = mn \).

(b) First note that \((e, e)\) is the identity of \( G \times H \) and \((a, b)^k = (a^k, b^k)\). So \((a, b)^k = (e, e)\) if and only of \( a^k = b^k = e \), which is equivalent to \( m|k \) and \( n|k \). In particular, \( m|\text{ord}((a, b)) \) and \( n|\text{ord}((a, b)) \). Write \( m = lm' \), \( n = ln' \) and \( k = \text{ord}((a, b)) \). As \( m|k \), \( k = tm = tlm' \) for some \( t \in \mathbb{Z} \). Then \( n = ln'|k = ltm' \), we see that \( n'|tm' \). But \( n' \) and \( m' \) are relatively prime, this implies \( n'|t \) and \( t = t'n' \). Therefore \( k = tlm' = t'n'lm' = t'n\frac{mn}{l} \). So we conclude that \( s := \frac{mn}{l} | \text{ord}((a, b)) \).

On the other hand, \( s = \frac{mn}{l} = m'n = mn' \). Therefore \( m|s \) and \( n|s \), and then \((a, b)^s = (e, e)\). Consequently \( \text{ord}((a, b))s = \frac{mn}{l} \). In summary, we have \( \frac{mn}{l} = \text{ord}((a, b)) \).

(c) If \( m \) and \( n \) are relatively prime then \( 1 = \gcd(m, n) \). By b), \( \text{ord}((a, b)) = mn \). Since \( |G \times H| = mn \), we see that \((a, b)\) generates \( G \times H \). Hence \( G \times H \) is cyclic.