Math 453, Midterm 1

Name: ____

This exam consists of 8 pages including this front page.

Ground Rules

1. No calculator is allowed.
2. Show your work for every problem unless otherwise stated.

Score		
1	15	
2	16	
3	20	
4	18	
5	18	
6	13	
Total	100	

Notations: In the following, $\mathbb{Z}$ denotes the group set of integers with addition, $\mathbb{Z}_n := \{\bar{0}, \bar{1}, \ldots, \overline{n-1}\}$ denote the group of $\mathbb{Z}$ modulo n, S_n denotes the group of all n-permutations. $\mathrm{GL}_n(R)$ denotes the group of invertible $n \times n$-matrices with entries in the set $\mathbb{R}$ of real numbers. G, H, K are always groups unless otherwise stated. Let $f : G \to H$ be a homomorphism. Then $\ker(f) := \{x \in G | f(x) = e\}$ denotes the kernel of f and $f(G) := \{y | y = f(x) \text{ for some } x \in G\}$ denotes the range of G.

1. The following are true/false questions. You don't have to justify your answers. Just write down either T or F in the table below. (3 points each).

(a) Let H_1, H_2 be subgroup of a group G, then $H_1 \cup H_2$ is a subgroup of G.

(b) Let $a_1, \ldots, a_n \in G$. Then $(a_1 \cdots a_n)^{-1} = a_1^{-1} \ldots a_n^{-1}$.

(c) Let $f : G \to H$ be a homomorphism. Then the range $f(G)$ is a normal subgroup of H.

(d) Let $\pi_1, \pi_2 \in S_n$ be permutations. If both π_1 and π_2 is odd then $\pi_1\pi_2$ is even.

(e) Any abelian group is cyclic.

	(a)	(b)	(c)	(d)	(e)
Answer	F	F	F	T	F

2. Multiple Choice, (4 points each):

(i) Which of the following relation $\sim$ is an equivalence relation on $\mathbb{R}^* = \{x \in \mathbb{R}, x \neq 0\}$.

(a) $x \sim y$ if $x \geq y$

(b) $x \sim y$ if $x = y^2$

(c) $x \sim y$ if $xy < 0$.

(d) $x \sim y$ if x/y is a rational number.

(e) $x \sim y$ if $\min\{x, y\} = x$.

The correct answer is D

(ii) Which of the following groups are isomorphic?

(a) $\mathbb{Z}_4$ and the group consisting of rotations of a square which keeps the square's position.

(b) $\mathbb{Z}_n$ and $\mathbb{R}^* = \{x \in \mathbb{R}, x \neq 0\}$ with multiplication.

(c) $\mathrm{GL}_n(R)$ and $\mathbb{Z}$.

(d) S_n and a cyclic group G with $|G| = n$.

(e) S_3 and the group consisting rotations of a square which keeps the square's position.

The correct answer is A.

(iii) Let $\pi := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \end{pmatrix} \in S_6$. Which of the following statement is NOT correct?

(a) π is a product of two disjoint cycles.
(b) π is even.
(c) The order of π is 3.
(d) The subgroup generated by π has 4 elements.
(e) π^2 is a generator of $\langle \pi \rangle$.

The correct answer is D

(iv) Let $G = \{e, a, b, b^2, ab, ab^2\}$ be a group whose generators a and b satisfy the equations,

$$a^2 = e, \ b^3 = e, \ ba = ab^2.$$

Which of the following statement is NOT correct:

(a) G can not be isomorphic to $\mathbb{Z}_n$ for some n.
(b) G is not abelian.
(c) G is isomorphic to a subgroup of S_6
(d) $\text{ord}(ab) = 3$.
(e) $\text{ord}(ab^2) = 2$.

The correct answer is D.

3. Let $G = \mathbb{Z}_{12} := \{\bar{0}, \bar{1}, \bar{2}, \cdots, \overline{11}\}$. (5 points each).

(a) Find the orders of $\bar{3}$ and $\bar{8}$.

(b) Find all i such that $\mathbb{Z}_{12} = \langle \bar{i} \rangle$.

(c) Find all subgroups of $\mathbb{Z}_{12}$.

(d) Let $\mathbb{R}^* := \{x \in \mathbb{R} | x \neq 0\}$ with multiplication. Can $\mathbb{Z}_{12}$ be isomorphic to a *subgroup* of $\mathbb{R}^*$. Why or why not?

Solutions

(a) $\text{ord}(\bar{3}) = 4$, $\text{ord}(\bar{8}) = 3$.

(b) Recall that a^k is generator of $\langle a \rangle$ with order n if and only if k and n are relatively prime. So $\bar{1}, \bar{5}, \bar{7}, \overline{11}$ has the property that $\mathbb{Z}_{12} = \langle \bar{i} \rangle$.

(c) Noe that subgroups of a cyclic group is still cyclic, we have the following subgroups: $\{0\}$, $\mathbb{Z}_{12}$, $\langle \bar{2} \rangle$, $\langle \bar{3} \rangle$, $\langle \bar{4} \rangle$, $\langle \bar{6} \rangle$.

(d) No, otherwise if $\mathbb{R}^*$ has a subgroup H which is isomomorphic to $\mathbb{Z}_{12}$. Then $\mathbb{R}^*$ contains an elements x with order 12. This means $x^{12} = 1$. That is $x = e^{\frac{2\pi i}{12}}$ which is not inside $\mathbb{R}$.

4. Let G be a group (6 points each).

(a) Show that $f : G \to G$ via $f(x) = x^{-1}$ is a isomorphism if and only if G is abelian.

(b) Let $f_a : G \to G$ be a function given by $f(x) = x^a$ with $a \in \mathbb{Z}$. Show that if G ia abelian then f_a is a homomorphism.

(c) Assume that G is abelian. When f_a is injective? prove your statement.

proof

(a) If G is abelian then

$$f(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = f(x)f(y).$$

So $f : G \to G$ is a homomorphism. To check that f is bijective, this is equivalent to check that for any $y \in G$, there exists a unique x so that $f(x) = y$. But $f(y^{-1}) = (y^{-1})^{-1} = y$ and the inverse of y is always unique. So f is bijiective and is an isomorphism.
Conversely, if f is an isomorphism. Then for any $x, y \in G$, we have

$$f(xy) = (xy)^{-1} = y^{-1}x^{-1} = f(x)f(y) = x^{-1}y^{-1}.$$

Note that $y^{-1}x^{-1} = x^{-1}y^{-1}$ is equivalent to that $xy = yx$. So G is abelian.

(b) Since G is abelain, we have $(xy)^a = x^a y^a$. Therefore, $f_a(xy) = (xy)^a = x^a y^a = f_a(x)f_a(y)$ and thus f_a is a homomorphism.

(c) f_a is injetcive if and only if $\ker(f_a) = \{e\}$. Since

$$\ker(f_a) = \{x | x^a = e\} = \{x \in G | \text{ord}(x) | a.\},$$

f_a is injective if and only if when $\text{ord}(x) | a$ then $x = e$.

5. Let G be a group, $a \in G$. Set $C_a := \{x \in G | ax = xa\}$. (6 points each)

(a) Show that C_a is a subgroup of G.

(b) Let $G = \mathrm{GL}_2(R)$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Find C_a.

(c) Is C_a always a normal subgroup? Prove or disprove your statement.

proof

(a) We need to check that C_a is closed under multiplication and inverse. For any $x, y \in C_a$, then

$$(xy)a = x(ya) = xay = (xa)y = axy.$$

and hence $xy \in C_a$. This prove that C_a is closed under multiplication. Since $xa = ax$, we have $xax^{-1} = axx^{-1} = a$, and then $x^{-1}xax^{-1} = x^{-1}a$. So $ax^{-1} = x^{-1}a$ and $x^{-1} \in C_a$. Thus C_a is closed under inverse and $C_a \subset G$ is a subgroup.

(b) Here $C_a = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$. This forces that $b = c = 0$. So $C_a = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} |a, d \in \mathbb{R} \right\}$.

(c) C_a is not always normal subgroup of G. In fact, in the above example $C_a = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} |a, d \in \mathbb{R} \right\}$ is not a normal subgroup of GL_2. Indeed let $x = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G$, $y \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in C_a$, we have

$$xyx^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \notin C_a.$$

6. Let G, H be groups. Recall that $G \times H = \{(a, x) | a \in G,\ x \in H\}$ together with operation $(a, x)(b, y) = (ab, xy)$ is a group. Now let $G = \langle a \rangle$ and $H = \langle b \rangle$ be cyclic groups with $\text{ord}(a) = m$ and $\text{ord}(b) = n$. Consider $G \times H = \langle a \rangle \times \langle b \rangle$.

(a) Show that $G \times H$ is abelian and $|G \times H| = mn$. (5 points)

(b) Let $l = \gcd(m, n)$. Show that $\text{ord}((a, b)) = \frac{mn}{l}$. (4 points)

(c) Show that if m, n are relatively prime then $G \times H$ is cyclic. (4 points)

proof:

(a) Since G and H are cyclic, G and H are abelian. Then for $(a, x), (b, y) \in G \times H$,

$$(a, x)(b, y) = (ab, xy) = (ba, yx) = (b, y)(a, x),$$

Thus $G \times H$ is abelian. Note that $|G \times H| = |G||H| = \text{ord}(a)\text{ord}(b) = mn$.

(b) First note that (e, e) is the identity of $G \times H$ and $(a, b)^k = (a^k, b^k)$. So $(a, b)^k = (e, e)$ if and only of $a^k = b^k = e$, which is equivalent to $m|k$ and $n|k$. In particular, $m|\text{ord}((a, b))$ and $n|\text{ord}((a, b))$. Write $m = lm'$, $n = ln'$ and $k = \text{ord}((a, b))$. As $m|k$, $k = tm = tlm'$ for some $t \in \mathbb{Z}$. Then $n = ln'|k = ltm'$, we see that $n'|tm'$. But n' and m' are relatively prime, this implies $n'|t$ and $t = t'n'$. Therefore $k = tlm' = t'n'lm' = t'\frac{mn}{l}$. So we conclude that $s := \frac{mn}{l}|\text{ord}((a, b))$.

On the other hand, $s = \frac{mn}{l} = m'n = mn'$. Therefore $m|s$ and $n|s$, and then $(a, b)^s = (e, e)$. Consequently $\text{ord}((a, b))|s = \frac{mn}{l}$. In summary, we have $\frac{mn}{l} = \text{ord}((a, b))$.

(c) If m and n are relatively prime then $1 = \gcd(m, n)$. By b), $\text{ord}((a, b)) = mn$. Since $|G \times H| = mn$, we see that (a, b) generates $G \times H$. Hence $G \times H$ is cyclic.