(1) Let $A$ be a Dedekind domain, $K$ its fraction field and $L/K$ a finite separable extension. Let $B$ be the integral closure of $A$ in $L$. Show that $B$ is also a Dedekind domain.

(2) Let $A$ be a domain. Assume that
(a) Every prime ideal of $A$ is maximal;
(b) Each ideal $a$ of $A$ admits unique factorization, namely, $a = p_1 \cdots p_m$ with $p_i \in \text{Spec}(A)$, and if $a = p_1 \cdots p_m = q_1 \cdots q_n$ with $q_i \in \text{Spec}(A)$ then $m = n$ and $p_i = q_i$ after reordering $q_i$.

Then show that $A$ is necessarily Dedekind.

(3) Let $A$ be a Dedekind domain and $a_1, a_2$ two ideals of $A$. Suppose that $a_1 = p_1^{r_1} \cdots p_m^{r_m}$ and $a_2 = p_1^{s_1} \cdots p_m^{s_m}$ with $p_i \in \text{Spec}(A)$ all distinct. Show the following
(a) $a_1 + a_2 = \prod_{i=1}^{m} p_i^\max\{r_i, s_i\}$ and $a_1 \cap a_2 = \prod_{i=1}^{m} p_i^\min\{r_i, s_i\}$
(b) if $a_1, a_2$ are relatively prime, i.e., $a_1 + a_2 = A$ then $A/a_1a_2 \simeq A/a_1 \times A/a_2$.

(4) Let $K = \mathbb{Q}(\sqrt{D})$ with $D$ a square free integer.
(a) Determine primes of $\mathbb{Z}$ that are ramified over $\mathcal{O}_K$ (Hint: Use Prop. 25 and HW 1 (3)).
(b) Let $p$ be an odd prime so that $p$ is unramified over $\mathcal{O}_K$.
Show that $p$ splits completely if and only if the equation $x^2 \equiv D \mod p$
has a solution. Or equivalently Legendre symbol $\left( \frac{D}{p} \right) = 1$.
(c) Discuss the factorization of $p = 2$ over $\mathcal{O}_K$.

(5) Let $K = \mathbb{Q}(\sqrt{q})$ with $q$ an odd prime. Let $\sigma$ denote the non-trivial element of $\text{Gal}(K/\mathbb{Q})$. Show that
(a) if $\left( \frac{q}{p} \right) = -1$ Then $\mathfrak{p} = p\mathcal{O}_K$ is still a prime of $\mathcal{O}_K$, i.e., $p$ is inert over $\mathcal{O}_K$.
(b) then $\sigma$ is the Frobenius at $\mathfrak{p}$, i.e., $(\mathfrak{p}, L/K) = \sigma$.
(c) show that $q^{\frac{p-1}{2}} \equiv -1 \mod p$ by using that $\sigma = (\mathfrak{p}, L/K)$. 


(6) Let $K = \mathbb{Q}(\alpha)$ with $\alpha^3 = 2$. Then one can show that $\mathcal{O}_K = \mathbb{Z}[\alpha]$. Determine the decomposition of prime for $p = 7, 29$. 