In the following, $K$ is always a number field. Recall Minkowski showed that any fractional $O_K$-ideal $b$ there exists an ideal $a$ such that $a \sim b$ and
\[ N(a) \leq \frac{n!}{n^n} \left( \frac{4}{\pi} \right)^n \sqrt{\Delta_K}. \]

(1) Show that the following quadratic field $\mathbb{Q} (\sqrt{D})$ has class number 1. $D = 5, -3, 2, -7$.

(2) Show that $\Delta_K$ goes to infinity when $n = [K : \mathbb{Q}]$ goes to infinity.

(3) (a) Let $a \subset O_K$ be an ideal. Suppose that $a^m = aO_K$ for $a \in O_K$. Show that $a$ become principal in the field $L = K(\sqrt[m]{a})$.
(b) Show there exists a finite extension $L$ of $K$ so that all ideal of $K$ become principal in $L$.

(4) Let $\zeta_m$ be primitive $m$-th root of unity. Show that $\frac{1 - \zeta^k}{1 - \zeta}$ for $(k, m) = 1$ are units of $O_{\mathbb{Q}(\zeta_m)}$. These units are called cyclotomic units.

(5) We have shown that if $p \nmid m$ then $\mathbb{Q}(\zeta_m)$ is unramified over $p$.
(a) Explicitly determine Frobenius at $p$. Namely, let $\chi$ denote the isomorphism $\chi : \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q}) \simeq (\mathbb{Z}/m\mathbb{Z})^\times$. Then what is the image of Frobenius at $p$ under $\chi$?
(b) Show that $p$ splits in $\mathbb{Q}(\zeta_m)$ if and only if $p \equiv 1 \ mod \ m$.

(6) Let $p$ be an odd prime. Then there exists a unique quadratic subfield $K \subset \mathbb{Q}(\zeta_p)$, which corresponds to the unique index 2 subgroup for $\text{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. Can you explicitly determine $K$? (Hint, study ramification of $K$).