MA690, Note 3/23

Recall: 1) V is de Rham if V is B_{dR}-admissible.

$$D := D_{dR}(V) := (V \otimes_{\mathbb{Q}_p} B_{dR})^{G_K}.$$

$$\mathrm{Fil}^i D = (V \otimes_{\mathbb{Q}_p} \mathrm{Fil}^i B_{dR})^{G_K} \subseteq D.$$

2): V is de Rham $\Rightarrow$ V is Hodge-Tate

$$\Rightarrow \bigoplus_{i=-\infty}^{+\infty} gr^i D \simeq D_{HT}(V) \quad \text{where } gr^i D = \mathrm{Fil}^i D / \mathrm{Fil}^{i+1} D$$

& $HT(V) := \{-i \mid gr^i D \neq \{0\}\}$.

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properties: ① If $0 \to V_1 \to V_2 \to V_3 \to 0$ is exact

then $0 \to \mathrm{Fil}^i D(V_1) \to \mathrm{Fil}^i D(V_2) \to \mathrm{Fil}^i D(V_3) \to 0$

②: $$\mathrm{Fil}^n D(V_1 \otimes V_2) = \sum_{i+j=n} \mathrm{Fil}^i D(V_1) \otimes_K \mathrm{Fil}^j D(V_2)$$

③ Let $V^* = \mathrm{HOM}_{\mathbb{Q}_p}(V, \mathbb{Q}_p)$ be the dual of V.

Then $D(V^*) \simeq \mathrm{Hom}_K(D, K)$, &

$$\mathrm{Fil}^n(D(V^*)) = \{f \in D(V^*) \mid f(\mathrm{Fil}^{-n+1} D) = 0\}.$$

proof: ① done. ③ an Exe from ②. It suffices to show ②:

First, it is clear that $\sum \mathrm{Fil}^i \otimes \mathrm{Fil}^j \subseteq \mathrm{Fil}^n$.

To prove "$=$", we can make induction n.

It is clear that "=" holds for $n << 0$ as $Fil^{-\infty} D = D$.

Now assume $\sum'_{i+j=n} Fil^i \otimes Fil^j = Fil^n$. Consider Fil^{n+1}.

Note that $\dfrac{Fil^n D(V_1 \otimes V_2)}{Fil^{n+1} D(V_1 \otimes V_2)} \simeq (\mathbb{C}_p(n) \otimes V_1 \otimes V_2)^{G_K}$

RHS is a K-v.s. with $\dim = \#\{(i,j) \mid i+j=n,\ \& \ -i \in HT(V_1),\ -j \in HT(V_2)\}$

Consider
$$LHS = \frac{Fil^n D(V_1 \otimes V_2)}{\sum'_{i+j=n+1} Fil^i D(V_1) \otimes Fil^j D(V_2)} = \frac{\sum'_{i+j=n} Fil^i \otimes Fil^j}{\sum_{i+j=n+1} Fil^i \otimes Fil^j}$$

$$= \bigoplus_{i+j=n} gr^i D(V_1) \otimes gr^j D(V_2)$$

Since $HT(V) = \{-i \mid gr^i D(V) \neq 0\}$.

$$\therefore \quad LHS \simeq \frac{Fil^n D(V_1 \otimes V_2)}{Fil^{n+1} D(V_1 \otimes V_2)}$$

$$\therefore \quad Fil^{n+1} D(V_1 \otimes V_2) = \sum_{i+j=n+1} Fil^i D(V_1) \otimes Fil^j D(V_2).$$

property (4): V is de Rham $\Leftrightarrow$ $V|_{G_{K'}}$ is de Rham for any finite ext. K'/K.

proof "$\Rightarrow$" clearly by standard admissible property.

"$\Leftarrow$": $D_{dR}(V|_{G_{K'}}) = (V \otimes_{\mathbb{Q}_p} B_{dR})^{G_{K'}}$ is a finite K'-v.s. with $\dim = \dim V$.

WLOG, by "$\Rightarrow$", we may replace K' by Galois closure so we assume K'/K is Galois. Then

$$Gal(K'/K) \curvearrowright D_{dR}(V|_{G_{K'}})$$

and $\quad D_{dR}(V) = D_{dR}(V|_{G_{K'}})^{Gal(K'/K)}$.

Then by Galois descent, $D_{dR}(V)$ has K-dim $= \dim V$

& then V is de Rham. $\square$

Question: Is HT rep. always de Rham?

Answer: No!

Example: Consider exact sequence

$$0 \to \mathbb{Q}_p \to V \longrightarrow \mathbb{Q}_p(1) \to 0$$

i.e $\quad V \sim \begin{pmatrix} 1 & * \\ 0 & \varepsilon_p \end{pmatrix} \quad * \in H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(-1))$.

Assume $* \neq 0$ exists. We claim V is HT.

Note

$$\begin{array}{c} 0 \to D_{HT}(\mathbb{Q}_p) \to D_{HT}(V) \longrightarrow D_{HT}(\mathbb{Q}_p(-1)) \longrightarrow H^1(\mathbb{Q}_p \otimes B_{HT}) \\ \qquad\qquad\qquad \| \\ \qquad\qquad\qquad \bigoplus_{i=-\infty}^{\infty} (\mathbb{C}_p(i) \otimes V)^G \end{array}$$

where $G = G_{\mathbb{Q}_p}$.

It suffices to show $D_{HT}(V)$ has dim 2.

First

$$(\mathbb{C}_p \otimes \mathbb{Q}_p)^G \hookrightarrow (V \otimes \mathbb{C}_p)^{G_K}.$$

secondly:

$$0 \to \overset{\overset{0}{\|}}{(\mathbb{Q}_p \otimes \mathbb{C}_p(1))^G} \to (V \otimes \mathbb{C}_p(1))^G \longrightarrow (\mathbb{Q}_p(-1) \otimes \mathbb{C}_p(1))^{G_K}$$

$$\to H^1(\mathbb{Q}_p \otimes \mathbb{C}_p(1))$$

Since $H^1(\mathbb{C}_p(i)) = 0$ by Tate Thm, $\forall i \neq 0$.

Then $\quad (V \otimes \mathbb{C}_p(1))^G \xrightarrow{\sim} (\mathbb{C}_p)^G$

$\therefore$

$$\dim D_H(T) = 2.$$

But V can not be de Rham. By monodromy Thm. any de Rham rep is pot. semi-stable. The classification of semi-stable rep. will show that only $V \in H^1(\mathbb{Q}_p(-1))$ must be trivial class. i.e $V \simeq \varepsilon_p \oplus 1$. so V can not be de Rham.

To see $H^1(\mathbb{Q}_p(-1)) \neq \{0\}$,

By Euler characteristic formula for Galois cohomology.

$$-1 = \chi(\mathbb{Q}_p(-1)) = h^0(\mathbb{Q}_p(-1)) + h^1(\mathbb{Q}_p(-1)) + h^2(\mathbb{Q}_p(-1))$$

$$h^0(\mathbb{Q}_p(-1)) = \dim (\mathbb{Q}_p(-1))^G = 0, \quad h^1(\mathbb{Q}_p(-1)) = \dim H^1(\mathbb{Q}_p(-1))$$

$$h^2(\mathbb{Q}_p(-1)) \overset{\text{Tate duality}}{=} (\mathbb{Q}_p(-1))^\vee(1) = \dim (\mathbb{Q}_p(2))^{G_K} = 0$$

$\therefore \ h^1(\mathbb{Q}_p(-1)) \neq 0$.

See Rubin 2000, For local Tate duality ($\mathbb{Q}_p$-version) & Euler char. formula.

Chap? crystalline & semi-stable rep.

I: Construct A_{cris}, $B_{st} \subseteq B_{dR}$ s.t A_{cris}, B_{st} has φ-structure.