MA690 Note 3/25

Aim: Construct B_{cris}, $B_{st} \subseteq B_{dR}$.

I: Ring A_{cris}:

Recall $A_{inf} = W(R)$ with $\theta: A_{inf} \twoheadrightarrow O_{\mathbb{C}_p}$.

$\ker\theta = \xi A_{inf}$. E.g, $\xi = [\underline{p}] - p$ &

$\underline{p} = (\sqrt[p^n]{p})_{n\geq 0} \in O^b_{\mathbb{C}_p} = R$. Set: $\gamma_n(\xi) = \frac{\xi^n}{n!}$

Set $A_{cris} := \left\{ \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!} \;\middle|\; a_n \in W(R),\ a_n \to 0 \text{ p-adically} \right\}$

$\subseteq B^+_{dR}$.

Remark: 1): $\forall x = \sum_{n=0}^{\infty} a_n \frac{\xi^n}{n!}$ is clearly in B^+_{dR}

But the expression $\sum a_n \frac{\xi^n}{n!}$ may not be unique.

$\gamma_n(\xi) = \frac{\xi^n}{n!}$ is called divided power, coming from crystalline cohomology

2): It is easy to check $A_{cris} \subseteq B^+_{dR}$ is a subring.

by using fact that $\gamma_n(\xi)\gamma_m(\xi) = \binom{m+n}{m}\gamma_{m+n}(\xi)$.

II: properties:

1): $Fil^i A_{cris} := A_{cris} \cap Fil^i B^+_{dR}$

$= \left\{ \sum_{n\geq i} a_n \frac{\xi^n}{n!} \;\middle|\; a_n \in W(R),\ a_n \to 0 \text{ p-adically} \right\}$.

2): $G_K \curvearrowright A_{cris} \supseteq Fil^i A_{cris}$ stabe.

because $\forall g \in G_K$, $g(\xi)$ is also generator of $\ker\theta$.

$\therefore$ $g(\xi) = \mu(g)\xi$ where $\mu(g) \in W(R)^\times$.

Hence $g(x) = \sum g(a_n)\mu(g)^n \frac{\xi^n}{n!} \in A_{cris}$.

3) Frobenius φ extends from $W(R)$ to A_{cris}.

proof: $\varphi(\xi) = \varphi([\underline{p}] - p) = [\underline{p}]^p - p$

$= (\xi + p)^p - p \in (\xi, p) A_{\inf}$.

$\qquad\qquad\qquad \| $

$\qquad\qquad \xi\alpha + p\beta$

$$\therefore \quad \varphi(x) = \sum_{n=0}^{\infty} \varphi(a_n) \frac{(\xi\alpha + p\beta)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \varphi(a_n) \left(\sum_{i=0}^{n} \gamma_i(\xi\alpha) \gamma_{n-i}(p\beta) \right)$$

where $\gamma_i(a) = \frac{a^i}{i!}$

$$= \sum_{i=0}^{\infty} \underline{\left(\sum_{n \geq i}^{\infty} \varphi(a_n) \gamma_{n-i}(p\beta) \right)} \gamma_i(\xi\alpha)$$

$\downarrow$

0 p-adically.

in $W(R)[\frac{1}{p}]$.

Remark ① $t = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([\varepsilon]-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(n-1)!\,([\varepsilon]-1)^n}{n!}$

as $(n-1)! \to 0$ p-adically. $\qquad \in A_{cris}$

$$\varphi(t) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{([\varepsilon]^p - 1)^n}{n} = \ln([\varepsilon]^p) = p \ln([\varepsilon])$$

$= pt$.

② In general, $\varphi(\mathrm{Fil}^i A_{cris}) \nsubseteq \mathrm{Fil}^i A_{cris}$ because $\varphi(\xi) \notin \mathrm{Fil}^1 A_{cris}$.

III topology of A_{cris}.

topo. of A_{cris} is p-adic topo.

claim: $W(R) \overset{i}{\hookrightarrow} A_{cris} \overset{j}{\hookrightarrow} B_{dR}^+$ are continuous maps.

proof. Note that $W(R)$ use (p, ξ)-topo. so if

If $x \to 0$ in $W(R)$, $x \in (\xi, p)^n$ for $n >> 0$.

Since $\xi^m = m!\, \gamma_m(\xi)$. & $m! \to 0$ p-adically,

$x \to 0$ in A_{cris}. Now if $x = \sum_{n=0}^{\infty} a_n \gamma_n(\xi) \to 0$

in A_{cris}. $\Leftrightarrow$ $a_n \to 0$ p-adically. (Here is ~~cop~~ gap here see it? :)).

Then $x \to 0$ in B_{dR}^+ for projective topo., recall that

$$B_{dR}^+ = \lim_{n \to \infty} W(R)[\tfrac{1}{p}] / \xi^n$$ and for each

$W(R)[\frac{1}{p}]/\xi^n$, we use topo. of $W(R)/\xi^n$ that induced from $W(R)$.

IV Construction of B_{st}^+.

Let $$u = \ln[p] := \ln \frac{[p]}{p} = \sum_{n=0}^{\infty} (-1)^{n-1} \frac{\left(\frac{[p]}{p} - 1\right)^n}{n} \in B_{dR}^+.$$

$$= \sum_{n=0}^{\infty} (-1)^{n-1} \frac{\xi^n}{p^n n} \in B_{dR}^+$$

u seems NOT in A_{cris}, as $v_p(p^n n) >> v_p(n!)$

But this is NOT easy to prove! will do this later.

Let $B_{st}^+ = A_{cris}[\frac{1}{p}][u] \subseteq B_{dR}^+$. write $B_{cris}^+ = A_{cris}[\frac{1}{p}]$

Set $B_{cris} = B_{cris}^+[\frac{1}{t}]$ & $B_{st} = B_{st}^+[\frac{1}{p}]$.

We will see that B_{cris}, B_{st} are period ring to define crystalline reps. & semi-stable reps.