Recall:
\[ \text{Al} = \{ x = \sum_{n=0}^{\infty} a_n 1_n (x) \mid a_n \in W(R), a_n \to 0 \text{ p-adiically} \} \]

\[ \text{BC} = \text{Al} \sqcup \frac{1}{p} T, \quad \text{BC}^+ = \text{Al} \sqcup \left[ \frac{1}{p}, u \right], \quad \text{BC}_e = \text{BC}^+ \left[ \frac{1}{e} \right] \]

where \[ u = \ln \left( \frac{1 + \frac{1}{p}}{p} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n^2} p^{-n} \in \text{BdR}^+ \]

Let \( k_0 \subseteq k \) be the sub-max. unramified ext. Then
\[ k_0 = W(k) \left[ \frac{1}{p} \right] \text{ where } k \text{ is the residue field of } K. \]

Note \( W(k) \subseteq W(k_0) \subseteq W(R) \).

Aim: Show that \( k_0 = (\text{BC}_e)^G \subseteq k = (\text{BdR})^G \)

To achieve this goal: Note \( \overline{k} \subseteq \text{BdR} \), we have to show:

\[ k_0 \subseteq \text{BdR} \quad \text{and} \quad k_0 \text{ has transcendental }/ \text{BC}_e \text{ where } \]

\( \text{Cons} \) is the fraction field of \( \text{Al} \).

Today, we try to do step 1:

**Lemma 1:** \( k_0 \otimes k_0 W(R) \left[ \frac{1}{p} \right] \to \text{BdR}^+ \) is injective.

\[ (a \otimes x) \rightarrow ax \]

**Proof:** It suffices to show \( O_k \otimes_{W(k)} W(R) \to \text{BdR}^+ \) is injective. Note \( O_k = W(k) \left[ \pi \right] \) where \( \pi \) is a uniformizer of \( O_k \). So \( W_k(R) = O_k \otimes_{W(k)} W(R) \)

\[ = \bigoplus_{i=0}^{e-1} W(R) \pi^i \quad \text{where } [k : k_0] = e. \]
Consider map: \( W_k(R) \to \text{Bd}^+ \to \text{C}_p \)

This map sends \( \sum_{i=0}^{e-1} \alpha_i \pi^i \to \sum_{i=0}^{e-1} \Theta(\alpha_i) \pi^i \) where \( \alpha_i \in W(R) \).

Still call this map \( \Theta \): so for any \( x \in W_k(R) \) which \( = 0 \in \text{Bd}^+ \), then \( x \in \ker \Theta \). Set \( \pi^b = (\prod_{i=0}^{e-1} \pi^i)^{1/n} \in \text{C}_p \).

Claim 1: \( \ker \Theta = \{ [\pi^b]^l - \pi \} W_k(R) \).

If claim holds, write \( u_\pi = [\pi^b]^l - \pi \), then \( x = u_\pi y \).

For map \( f: W_k(R) \to \text{Bd}^+ \), \& \( x \in \ker f \).

\[ 0 = f(x) = f(u_\pi y) = u_\pi f(y) \Rightarrow f(y) = 0 \]

as \( \text{Bd}^+ \) is an integral domain. \( \therefore x = \bigcap_{n=0}^{\infty} u_\pi^n W_k(R) \)

If we can show claim 2: \( \bigcap_{n=0}^{\infty} u_\pi^n W_k(R) = \{0\} \) then \( x = 0 \)

\& Lemma 1 follows.

Proof of Claim 1: Let \( E(x) \) be Eisenstein poly of \( \pi \).

Then \( E(x) \in W(R)[x] \) \& \( E(\pi) = 0 \).

Consider \( \eta = E([\pi^b]^l) \in W(R) \). We see that \( \Theta(\eta) = E(\pi) = 0 \).

Note that \( \nu_k(\eta \mod p) = \nu_k([\pi^b]^l \mod p) = \nu_k([\pi^b]^l) = \nu_p(\pi^l) = 1 \). \( \eta \) is a generator of \( \ker \Theta \).

(In general, \( \# \in \ker \Theta, \& \nu_k(\eta \mod p) = 1 \) then \( \ker \Theta = \{0\} \).

This proof can be seen from the proof that \( \ker \Theta = \{x\} \).

Also \( \eta = E([\pi^b]^l) = \prod_{i=0}^{e-1} (\pi^b I - \pi^i) \) where \( \pi^i \) are all roots of \( E(x) \) \& \( \pi_0 = \pi \). \( \therefore u_\pi \mid \eta \) in \( W_k(R) \).

Now let \( x \in \ker \Theta \), \( x = \sum_{i=0}^{e-1} a_i \pi^i \) with \( a_i \in W(R) \)

Then \( x = \sum a_i [\pi^b]^i + \sum a_i (\pi^i - [\pi^b]^i) \)
It suffices to show first term $\sum_i a_i [\pi^b J]^i$ must be $u_{\pi} y$. but $\Theta(x) = \Theta(\sum_i a_i [\pi^b J]^i) = c$.

Thus $z = \sum_i a_i [\pi^b J]^i \in \ker \Theta | \text{w}(R) \Rightarrow z = \eta z'$, $z' \in \text{w}(\Theta)$ as $u_{\pi} | \eta$. $x = u_{\pi} z''$ with $z'' \in \text{w}(R)$.

Proof of claim 2: By binomial expansion, $u_{\pi}^n = \sum_{i=0}^{e-1} b_i(n) \pi^i$ with $b_i(n) \in \text{w}(R)$. It is easy to check $b_i(n) \to 0$ via $(\pi^b J, p)$-topology, which is -topology of $\text{w}(R)$.

Now we try to prove $K \otimes_{k_0} B_{crs} \rightarrow B_{dr}$.

First, since $p, t \in B_{dr}$ non-zero & $B_{dr}$ is an integral domain, it suffices to show $f: K \otimes_{w(R)} A_{cr} \to B_{dr}$ clearly $K \otimes_{w(R)} A_{cr} \subseteq \sum_{i=0}^{e-1} A_{cr} \pi^i$. pick $x \in \text{ker} f$.

$x = \sum_{i=0}^{e-1} a_j \pi^i$ with $a_j \in A_{cr}$. WLOG, we may assume that $p \nmid a_0$. In fact if $p | a_0$, then $x = \pi y$ with $y \in K \otimes_{w(R)} A_{cr}$, hence $y \in \text{ker} f$ & we may replace $x$ by $y$.

Now we write $x = \sum_{i=0}^{e-1} \sum_{i=0}^{\infty} a_{ij} f_i(x) \pi^i$.

recall that $a_{ij} \to 0$ p-adically when $i \to \infty$.
Recall that \( \eta = E \left( \left[ \pi^k \right] \right) \) is another generator of \( \ker \theta \mid_{W(R)} \). So we may let \( y = \eta \). By using \( \eta = \sum_{i=0}^{\infty} b_i \pi_i \pi^e \) with \( b_i \in \mathcal{O}_k \), we may rewrite

\[
x = \sum_{i=0}^{\infty} \frac{c_i \pi^i}{e(i)!} \quad \text{where} \quad c_i \in W_k(R), \ c_i \to 0
\]

& \quad e(i) = \left[ \frac{i}{e} \right] 7

Pick \( m > 0 \) so that \( 1) \ p \mid c_i, \ \forall i > m, \ 2) \ v_p(e(m+i)) \to v_p(e(m)) \)

\( f(x) = 0 \implies L = \sum_{i=0}^{m} c_i \frac{U_i}{e(i)!} = - \sum_{i=m+1}^{\infty} \)

(We calculate this in \( B^+ \mathcal{O}_k \), we drop \( f \) for simplicity.)

Let \( L = \text{max} \left\{ v_p(e(i)) \right\} \). Then \( p^L L \in U^{m+1}_\pi \mathcal{B}^+ \mathcal{O}_k \)

Since \( p^L L \in W_k(R) \cap U^{m+1}_\pi \mathcal{B}^+ \mathcal{O}_k \). As \( \ker \theta = U_\pi W_k(R) \), we have \( p^L L \in U^{m+1}_\pi W_k(R) \).

\[
L = \frac{U^{m+1}_\pi}{p^2} \cdot Z, \quad Z \in W_k(R).
\]

\[
x = \frac{U^{m+1}_\pi}{p^2} + \sum_{i=m+1}^{\infty} c_i \frac{U_i}{e(i)!}
\]

Note \( \frac{U^{m+1}_\pi}{p^2} \in \mathfrak{p} \left( \mathcal{O}_k \otimes_{\mathcal{O}(k)} \text{Acm} \right) \) & \( p \mid c_i \).

\( \implies x \in \mathfrak{p} \left( \mathcal{O}_k \otimes_{\mathcal{O}(k)} \text{Acm} \right) \) contradicts to that \( p \nmid a_0 \).