MA690, 3/27

Recall: $A_{cris} = \{ x = \sum_{n=0}^{\infty} a_n \gamma_n(\xi) \mid a_n \in W(R),\ a_n \to 0 \text{ p-adically} \}$

$B_{cris} = A_{cris}[\frac{1}{p}, \frac{1}{t}]$, $B_{st}^{+} = A_{cris}[\frac{1}{p}, u]$, $B_{st} = B_{st}^{+}[\frac{1}{t}]$

where $u = \ln\left(\frac{[P]}{p}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\xi^n}{p^n n} \in B_{dR}^{+}$.

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Let $K_0 \subseteq K$ be the sub-max. unramified ext. Then $K_0 = W(k)[\frac{1}{p}]$ where k is the residue field of K.

Note $W(k) \subseteq W(\bar{k}) \subseteq W(R)$.

Aim: show that $K_0 = (B_{st})^G \subseteq K = (B_{dR})^G$

To achieve this goal: Note $\bar{K} \subseteq B_{dR}$, we have to show

$$K \otimes_{K_0} B_{st} \hookrightarrow B_{dR} \quad (\text{check this is enough.})$$

The proof will be 2 steps: 1): $K \otimes_{K_0} B_{cris} \hookrightarrow B_{dR}$

2) u is transcendental $/C_{cris}$ where C_{cris} is the fraction field of A_{cris}.

Today, we try to do step 1:

Lemma 1. $K \otimes_{K_0} W(R)[\frac{1}{p}] \longrightarrow B_{dR}$ is injective.

$$(a \otimes x) \longrightarrow ax$$

proof: It suffices to show $O_K \otimes_{W(k)} W(R) \longrightarrow B_{dR}^{+}$ is injective. Note $O_K = W(k)[\pi]$ where π is a uniformizer of O_K. so $W_K(R) := O_K \otimes_{W(k)} W(R)$

$$\simeq \bigoplus_{i=0}^{e-1} W(R)\pi^i \quad \text{where } [K:K_0] = e.$$

consider map: $W_K(R) \longrightarrow B_{dR}^+ \xrightarrow{\theta} \mathbb{C}_p$

This map sends $\sum_{i=0}^{e-1} a_i \pi^i \longrightarrow \sum_{i=0}^{e-1} \theta(a_i)\pi^i$ where $a_i \in W(R)$.

still call this map θ: so for any $x \in W_K(R)$ which $= 0 \in B_{dR}^+$

then $x \in \operatorname{Ker}\theta$. Set $\pi^b = (\sqrt[p^n]{\pi})_{n\geq 0} \in \mathcal{O}_{\mathbb{C}_p}^b$

Claim 1: $\operatorname{Ker}\theta = ([\pi^b] - \pi)\, W_K(R)$.

If claim holds, write $u_\pi = [\pi^b] - \pi$, then $x = u_\pi y$.

For map $f: W_K(R) \longrightarrow B_{dR}^+$. & $x \in \operatorname{Ker} f$.

$0 = f(x) = f(u_\pi y) = u_\pi f(y) \Rightarrow f(y) = 0$

as B_{dR}^+ is an integral domain. $\therefore$ $x = \bigcap_{n=0}^{\infty} u_\pi^n W_K(R)$

If we can show claim 2: $\bigcap_{n=0}^{\infty} u_\pi^n W_K(R) = \{0\}$ then $x = 0$

& Lemma 1 follows.

proof of claim 1: Let $E(x)$ be Eisenstein poly. of π.

Then $E(x) \in W(k)[x]$ & $E(\pi) = 0$.

consider $\eta = E([\pi^b]) \in W(R)$. We see that $\theta(\eta) = E(\pi) = 0$.

Note that $v_R(\eta \,(\mathrm{mod}\, p)) = v_R([\pi^b]^e \bmod p) = v_R((\pi^b)^e)$

$= v_p(\pi^e) = 1$. η is a generator of $\operatorname{Ker}\theta$.

(In general, if $z \in \operatorname{ker}\theta$, & $v_R(z \bmod p) = 1$ then $\operatorname{Ker}\theta = z\,W(R)$
This proof can be seen from the proof that $\operatorname{Ker}\theta = (\xi)$)

Also $\eta = E([\pi^b]) = \prod_{i=0}^{e} ([\pi^b] - \pi_i)$ where

π_i are all roots of $E(x)$ & $\pi_0 = \pi$. $\therefore$ $u_\pi \mid \eta$ in $W_K(R)$

Now let $x \in \operatorname{ker}\theta$, $x = \sum_{i=0}^{e-1} a_i \pi^i$ with $a_i \in W(R)$

Then $x = \sum a_i [\pi^b]^i + \sum a_i (\pi^i - [\pi^b]^i)$

It suffices to show first term $\sum_i a_i [\pi^b]^i$ must be $u_\pi y$. but $\theta(x) = \theta(\sum_i a_i [\pi^b]^i) = 0$.

$\therefore\ z = \sum_i a_i [\pi^b]^i \in \ker \theta|_{W(R)}$. $\Rightarrow z = \eta \cdot z'$, $z' \in W(R)$

as $u_\pi | \eta$. $x = u_\pi \cdot z''$ with $z'' \in W_K(R)$.

proof of claim 2: By binomial expansion, $u_\pi^n = \sum_{i=0}^{e-1} b_i(n) \pi^i$ with $b_i(n) \in W(R)$. It is easy to check $b_i(n) \longrightarrow 0$ via $([\pi^b], p)$-topology. which is topo. of $W(R)$.

Now we try to prove $K \otimes_{K_0} B_{cris} \hookrightarrow B_{dR}$.

First, since $p, t \in B_{dR}$ non zero & B_{dR} is an integral domain, it suffices to show $f: O_K \otimes_{W(k)} A_{cris} \hookrightarrow B_{dR}^+$

clearly $O_K \otimes_{W(k)} A_{cris} \subset \bigoplus_{i=0}^{e-1} A_{cris} \pi^i$. pick $x \in \ker f$

$x = \sum_{j=0}^{e-1} a_j \pi^j$ with $a_j \in A_{cris}$. WLOG, we may assume that $p \nmid a_0$. In fact if $p | a_0$, then $x = \pi y$ with $y \in O_K \otimes_{W(k)} A_{cris}$, hence $y \in \ker f$ & we may replace x by y.

Now we write $x = \sum_{j=0}^{e-1} \sum_{i=0}^{\infty} a_{ij} \gamma_i(\xi) \pi^j$

recall that $a_{ij} \longrightarrow 0$ p-adically when $i \longrightarrow \infty$

Recall that $\eta = E([\pi^b])$ is another generator of $\mathrm{Ker}\,\Theta|_{W(R)}$

so we may let $\xi = \eta$. By using $\eta = \sum_{i=0}^{e-1} b_i u_\pi^i \pi^{e-i}$

with $b_i \in \mathcal{O}_K$, we may rewrite

$$x = \sum_{i=0}^{\infty}{}' c_i \frac{u_\pi^i}{e(i)!} \quad \text{where } c_i \in W_K(R),\ c_i \to 0$$

& $e(i) = \lceil \frac{i}{e} \rceil$.

pick $m \gg 0$. s.t 1) $p \mid c_i$, $\forall i > m$, 2) $v_p(e(m+1)!) > v_p(e(i)!)$ for all $i \le m$.

$$f(x) = 0 \Rightarrow L = \sum_{i=0}^{m}{}' c_i \frac{u_\pi^i}{e(i)!} = -\sum_{i=m+1}{}' \text{——}$$

(we ~~计~~ calculate this in B_{dR}^+, we drop f for simplicity)

$\therefore$ Let $l = \max\{v_p(e(i)!)\}$. Then $p^l L \in u_\pi^{m+1} B_{dR}^+$

Since $p^l L \in W_K(R) \cap u_\pi^{m+1} B_{dR}^+$. As $\ker\Theta := u_\pi W_K(R)$,

we have $p^l L \in u_\pi^{m+1} W_K(R)$.

$$\therefore \quad L = \frac{u_\pi^{m+1}}{p^l} \cdot z, \quad z \in W_K(R).$$

$$\therefore \quad x = \frac{u_\pi^{m+1}}{p^l} + \sum_{i=m+1}^{\infty} c_i \frac{u_\pi^i}{e(i)!}$$

Note $\frac{u_\pi^{m+1}}{p^l} \in p(\mathcal{O}_K \otimes_{W(k)} A_{cris})$ & $p \mid c_i$.

$\Rightarrow$ $x \in p(\mathcal{O}_K \otimes_{W(k)} A_{cris})$ contradicts to that

$p \nmid a_0$.