Recall: Have proved $K \otimes_{K_0} B_{cris} \hookrightarrow B_{dR}$.

For today, aim to prove $u = \ln\left(\frac{[P]}{p}\right)$ is transcendental / $C_{cris} = \mathrm{Frac}(B_{cris})$.

If aim can be proved then $K \otimes_{K_0} B_{st} \simeq (K \otimes_{K_0} B_{cris})[u]$ injects to B_{dR}. To see this, Note $\mathrm{Frac}(K \otimes_{K_0} B_{st})$ is algebraic over C_{cris}. The aim shows that u is transcendental / $K \otimes_{K_0} B_{cris}$

$\therefore$ $K \otimes_{K_0} B_{st} \hookrightarrow B_{dR}$.

Corollary: $K_0 = (B_{st})^{G_K} = (B_{cris})^{G_K}$.

Lemma 1: $u \notin C_{cris}$.

Indeed Lemma 1 implies aim.

proof of this implication:

$\forall g \in G_K$, $g[\underline{P}] = [\underline{P}][\underline{\varepsilon}^{\sigma(g)}]$. $\underline{\varepsilon} = (\zeta_{p^n})_{n \geq 0} \in \mathcal{O}_{\mathbb{C}_p}^{b} = R$

where $\underline{\varepsilon}^{\sigma(g)} \in H^1(G_K, R^\times)$.

$\therefore$ $g(u) = \ln\left(\frac{[\underline{\varepsilon}]^{\sigma(g)}[\underline{P}]}{p}\right) = u + \sigma(g)\ln([\underline{\varepsilon}])$
$= u + \sigma(g)\, t$.

pick $f(X) \in C_{cris}[X]$ the monic poly. s.t. $f(u) = 0$.

write $f(X) = \sum_{i=0}^{d} c_i X^i$, with $c_i \in C_{cris}$.

$$0 = g(f(u)) = \sum_{i=0}^{d} g(c_i)(g(u))^i$$
$$= \sum_{i=0}^{d} g(c_i)(u + \sigma(g)t)^i$$
$$= f(u) = \sum_{i=0}^{d} c_i u^i$$

WLOG, we may assume that f is minimal. & $c_d = 1$.

$\therefore \quad d\,\sigma(g)t + g(c_{d-1}) = c_{d-1}$.

$\therefore \quad g(c_{d-1} + du) = c_{d-1} + du \qquad \forall g \in G_K$.

$c_{d-1} + du \in (B_{dR})^{G_K} = K$. If we pick $K = \mathbb{Q}_p$.

$\therefore \quad c_{d-1} + du \in B_{cris}$ & $u \in C_{cris}$. Contradiction.

$\square$.

It suffices to show Lemma 1:

Let $\beta = -\frac{\xi}{p}$. Then both $\xi, \beta \in \mathrm{Fil}^1 B_{dR} \setminus \mathrm{Fil}^2 B_{dR}$.

& $u = \sum_{n=1}^{\infty} \frac{\beta^n}{n} \in B_{dR}^+$. Set $S = W(R)[\![\beta]\!] \subseteq B_{dR}^+$

Check $A_{cris} \subseteq S$. So $C_{cris} \subseteq \mathrm{Frac}(S)$. To prove $u \notin C_{cris}$, it suffices to show $\forall \alpha \in S,\ \alpha \neq 0,\ \alpha u \notin S$.

WLOG, we assume $\alpha \notin pS$ & $p^r \alpha u \in S$, try to prove a contradiction.

For $a \in W(R)$, if $\theta(a) \in p\,\mathcal{O}_{\mathbb{C}_p}$. then $a = pb + c\xi$ with $b, c \in W(R)$. $\therefore \ a = p(b + c\beta) \in pS$.

So if $\alpha \notin pS$ then $\exists\, i \geq 0$ s.t

$$\alpha = \underbrace{p\Big(\sum_{0\le n<i}' b_n \beta^n\Big)}_{A = pA'} + \underbrace{\sum_{n\ge i}' b_n \beta^n}_{B}, \quad b_n \in W(R)$$

& $\theta(b_n) \notin p\mathcal{O}_{\mathbb{C}_p}$.

Pick $j > r$. s.t. $p^j > i$ & $j-1 > r$ $\Rightarrow$ $\boxed{\alpha p^{j-1}\Big(u = \sum' \frac{\beta^n}{n}\Big) \in S.}$

Note that $p^{j-1}\sum_{n<p^j}' \frac{\beta^n}{n} \in S$, because $\frac{p^{j-1}}{n} \in \mathbb{Z}_p$ when $n < p^j$

$$\therefore \quad \alpha\, p^{j-1} \sum_{n\ge p^j}' \frac{\beta^n}{n} \in S.$$

$$\| $$

$$(A+B)\Big(\frac{p^{j-1}\beta^{p^j}}{p^j} + \sum_{n>p^j}' \frac{p^{j-1}\beta^n}{n}\Big) \in S.$$

Since $A = pA'$, $\quad A\Big(\sum_{p^j\le n<p^{j+1}}' \frac{p^{j-1}\beta^n}{n}\Big) \in S.$

$$\therefore \quad B\frac{\beta^{p^j}}{p} + A\sum_{n\ge p^{j+1}}' \frac{p^{j-1}\beta^n}{n} + B\sum_{n>p^j}' \in S.$$

$$B\frac{\beta^{p^j}}{p} = b_i \frac{\beta^{p^j+i}}{p} + \sum_{n\ge i+1}' \frac{\beta^{p^j}}{p} b_n \beta^n$$

$$\therefore \quad b_i \frac{\beta^{p^j+i}}{p} \in S + \mathrm{Fil}^{p^{j+1}} B_{dR} + \mathrm{Fil}^{p^j+i+1} B_{dR}.$$

$$\subseteq S + \mathrm{Fil}^{p^j+i+1} B_{dR}$$

Note $\alpha \in S \cap \mathrm{Fil}^m B_{dR}$, $\Rightarrow \alpha = \sum_{n\ge m}' a_n \beta^n$, $a_n \in W(R)$.

Note define $\Theta_m : \mathrm{Fil}^m B_{dR}^+ \longrightarrow \mathbb{C}_p$

$$x \longmapsto \Theta\left(\frac{x}{\beta^m}\right)..$$

Let $m = p^j + i$, we have $b_i \frac{\beta^{p^j+i}}{p} = \alpha + \beta$

with $\alpha \in S \cap \mathrm{Fil}^m B_{dR}$. & $\beta \in \mathrm{Fil}^{m+1} B_{dR}$.

$\therefore \Theta_m(\alpha) = \Theta(a_m) \in \mathcal{O}_{\mathbb{C}_p}$. $\Theta_m(\beta) = 0$.

$\therefore \Theta_m\left(b_i \frac{\beta^m}{p}\right) \in \mathcal{O}_{\mathbb{C}_p}$.

but $\Theta_m\left(\frac{b_i \beta^m}{p}\right) = \Theta\left(\frac{b_i}{p}\right) \notin \mathcal{O}_{\mathbb{C}_p}$.

Contradiction.