MA690 Apr. 3rd.

Recall:
$$K \otimes_{K_0} B_{st} \hookrightarrow B_{dR}.$$

Then
$$K \subseteq (K \otimes_{K_0} B_{st})^{G_K} = (B_{dR})^{G_K} = K.$$
$$\| $$
$$K \otimes_{K_0} (B_{st})^{G_K}$$

$\therefore$ $(B_{st})^{G_K} = K_0 = (B_{cris})^{G_K} \supseteq (W(R)[\frac{1}{p}])^{G_K} \supseteq K_0.$

---

prop 1: B_{cris}, B_{st} is $(\mathbb{Q}_p, G_K)$-regular.

proof: Both B_{cris}, B_{st} are domains & $\mathbb{Q}_p \subseteq B_{cris}^{G_K} = B_{st}^{G_K} = K_0$.

$\therefore$ It suffices to show $\forall x \in B_{st}$, with $x \neq 0$.

If $g(x) = \eta(g) \cdot x$ with $\eta(g) \in \mathbb{Q}_p^\times$ then $x \in B_{st}^\times$.

Since $t^i \in B_{cris}^\times$, after replacing x via suitable $x t^i$, we may assume $x \in \mathrm{Fil}^0 B_{dR} \setminus \mathrm{Fil}^1 B_{dR}$.

Let $y = \theta(x) \in \mathbb{C}_p$. $\Rightarrow$ $g(y) = \eta(g)\, y$.

$\therefore$ $y \in (\mathbb{C}_p(\eta^{-1}))^{G_K}$. By Tate's thm,

$\Rightarrow$ $\eta(I_K)$ is finite. $\therefore$ $\exists$ K'/K finite ramified,

$\eta|_{G_{K'}} = \mathrm{Id}$. $\therefore$ $y \in (\mathbb{C}_p(\eta^{-1}))^{G_{K'}} = K'$.

Since $\overline{K} \subseteq B_{dR}^+$, we may assume that $y \in \overline{K} \subseteq B_{dR}^+$

$\therefore$ $x - y = t^m z$. with $z \notin \mathrm{Fil}^1 B_{dR}^+$.

$g(x-y) = g(t^m z) = \eta(g)(x-y) = \eta(g)\, t^m z.$

$\therefore \quad \varepsilon^m(g)\, t^m Z = \eta(g)\, t^m Z.$

$\therefore \quad Z \in \left(\mathbb{C}_p(\varepsilon^m \eta^{-1})\right)^{G_K}$ for $m \geq 1$.

$\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad \{0\}.$

$\therefore \quad x = y \in K'. \implies x \in K' \cap B_{st}$

Since $K' \otimes_{K_0'} B_{st} \hookrightarrow B_{dR}^+ . \Rightarrow x \in K_0' \subseteq B_{st}^\times$.

· Similar argument applies to B_{cris}.

Def: V is crystalline (semi-stable, or log-crystalline) if V is B_{cris} (resp. B_{st}) - admissible.

structure of D_{cris} & D_{st}.

1) Frobenius: Let φ denote Frobenius $/W(R)$. Since φ extends to A_{cris}, it extends to B_{cris}. Commutes with G_K.

set $\varphi(u) = pu$. because $u = \ln\left(\frac{[P]}{p}\right)$.

Check: $g\varphi = \varphi g, \quad \forall g \in G_K$ on B_{st}.

as $g(u) = u + \sigma(g)t$.

So: $\exists\ \varphi_D$ on $D_{st}(V) = (V \otimes_{\mathbb{Q}_p} B_{st})^{G_K}$

s.t: ①: $D_{st}(V)$ is finite K_0-v.s. $(\dim_{K_0} D \leq \dim_{\mathbb{Q}_p} V)$.

②: $\varphi_D : D_{st}(V) \longrightarrow D_{st}(V)$ is φ-semi-linear.

i.e $\varphi_D(ax) = \varphi_{K_0}(a)\, \varphi_D(x)$

$\forall a \in K_0, \quad x \in D := D_{st}(V).$

2) Filtration:

Assume V is semi-stable.

Lemma: $K \otimes_{K_0} D_{st}(V) = D_{dR}(V)$.

In particular, if V is semi-stable then it is deRham.

proof:

$$\begin{array}{ccc} (K \otimes_{K_0} B_{st} \otimes_{\mathbb{Q}_p} V)^{G_K} & \subseteq & (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_K} \\ \| & & \| \\ K \otimes_{K_0} (B_{st} \otimes_{\mathbb{Q}_p} V)^{G_K} & & D_{dR}(V) \\ \| & & \\ K \otimes_{K_0} D_{st}(V) & & \end{array}$$

since V is semi-stable, $D_{st}(V)$ has K_0-dim d, $\Longrightarrow$

$$K \otimes_{K_0} D_{st}(V) = D_{dR}(V).$$

Exe: V is crystalline $\Longrightarrow$ V is semi-stable.

In this case, $D_{cris}(V) = D_{st}(V)$.

Therefore, when V is semi-stable, write $D_K = K \otimes_{K_0} D_{st}(V)$.

$\exists \ \mathrm{Fil}^i D_K := \mathrm{Fil}^i D_{dR}(V). \subseteq D_K$.

3): Monodromy operator: on $B_{st} = B_{cris}[u]$.

$\forall \ x \in B_{st}$, write $x = f(u) \in B_{cris}[u]$.

define, $N(x) = \dfrac{df}{du}$.

properties: ① $B_{cris} = (B_{st})^{N=0}$.

② $N\varphi = p\varphi N$.

③ G_K action commutes with N.

All these can be reduced to check just for u^m because N is additive.

$\therefore \quad \exists\, N_D: \quad D_{st} \longrightarrow D_{st} \equiv (V \otimes_{\mathbb{Q}_p} B_{st})^{G_K}$ $\circlearrowleft N$.

so that: ① N_D is K_0-linear.

② N_D is nilpotent.

③ $p\varphi N = N\varphi$.

We will see concrete examples Next time.