Section 1.1

Direction fields.

Differential equations are equations containing derivatives. In these equations, the unknown is a <u>function</u>.

<u>Ex</u>: Find the function $y(t)$ such that:

$$(*) \quad y'(t) = 2y(t)$$

Note: We are familiar with equations for numbers. For example: Find x such that $x^2 = 2x$. We have $x^2 - 2x = 0$, that is, $x(x-2) = 0$, and $x_1 = 0$, $x_2 = 2$.

Trivially, we note that the function $y(t) \equiv 0$ solves equation $(*)$. It is the <u>function</u> identically zero (not the number 0).

Since the exponential is a function whose derivative is again the same function, we see that:

$$y(t) = e^{2t} \text{ solves } (*).$$

Indeed:
$$y'(t) = 2e^{2t} = 2y(t), \text{ for every } t.$$

But:

$$y(t) = 2e^{2t} \text{ also solves } (*),$$

because:

$$y'(t) = 4e^{2t} = 2(2e^{2t}) = 2y(t),$$

for every t.

Actually, there is a family of solutions: any function of the form:

$$y(t) = ce^{2t}, \quad c \text{ any real number},$$

is a solution of $(*)$ because if we plug $y(t)$ in $(*)$, the equation holds:

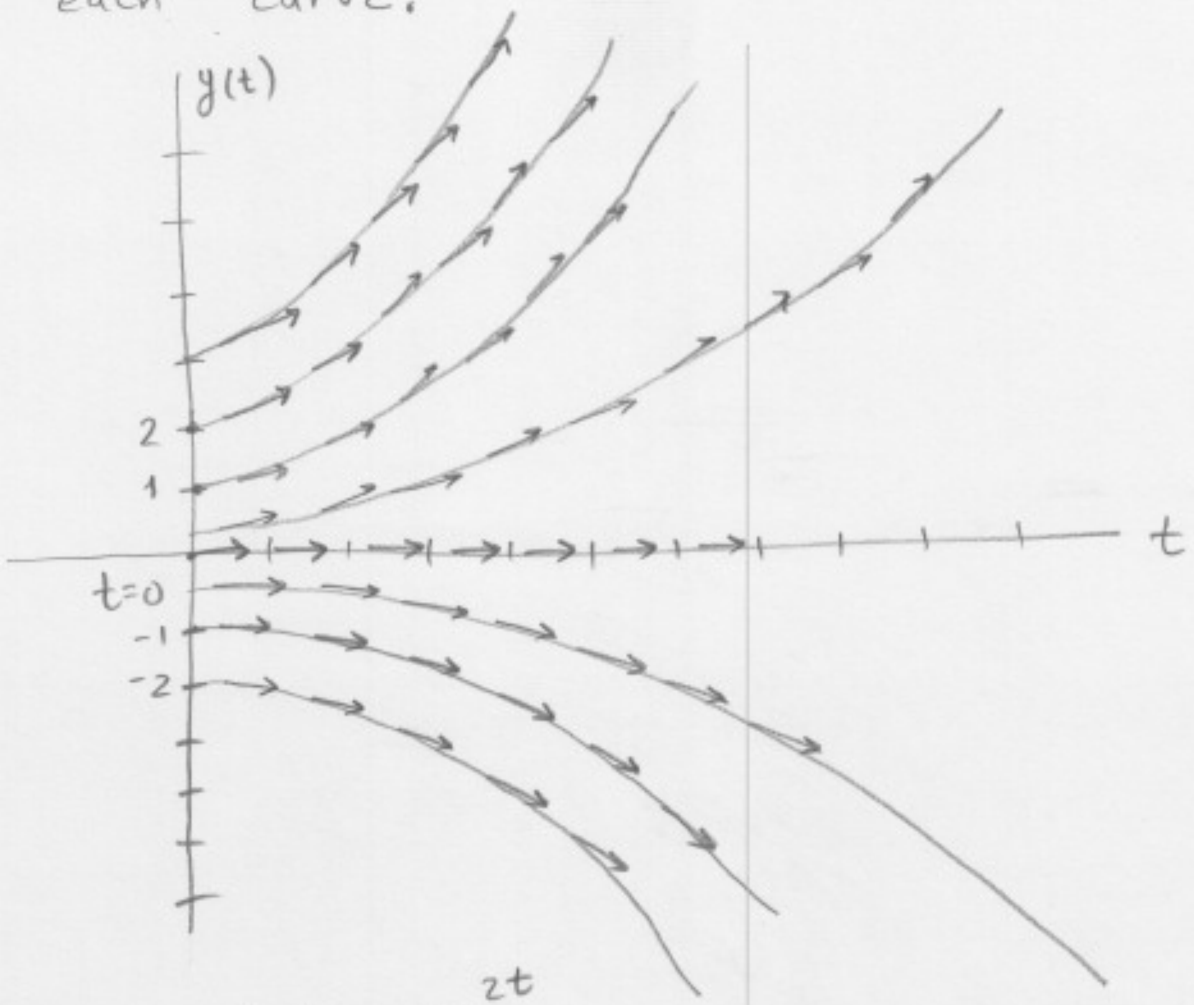
$$y'(t) = 2ce^{2t} = 2(ce^{2t}) = 2y(t), \text{ for every } t$$

Conclusion: We have found an infinite number of solutions to $y' = 2y$. The solutions form a family:

$$y(t) = ce^{2t}, \quad c \text{ any real number}.$$

Question: Are these the only solutions of $(*)$. Could we find another function (that is not of the form $y = ce^{2t}$) that solves $(*)$? The answer is no, but this needs to be proven mathematically.

We can now draw this family of solutions together with tangent vectors to each curve.



$y(t) = ce^{2t}$

$y(0) = ce^{2(0)} = C$, this is the initial condition, the value of y at $t=0$.

$C=1 \Rightarrow y = e^{2t}$

$C=2 \Rightarrow y = 2e^{2t}$

$C=0 \Rightarrow y(t) \equiv 0$, for every t

$C=-1 \Rightarrow y = -e^{2t}$

$C=-2 \Rightarrow y = -2e^{2t}$

<u>Note:</u> Two solution curves never intersect, this fact of differential equations needs to be proved mathematically.

Direction field

A direction field is the graph t versus y with all the arrows that are tangent to the solution curves.

If we only have the arrows, we can draw the solution curves, starting at some initial condition $y(0)$, and following the direction given by the arrows. From the direction field corresponding to $y' = 2y$ we see the following:

$$\lim_{t\to\infty} y(t) = \infty, \quad \text{if } y(0) > 0$$

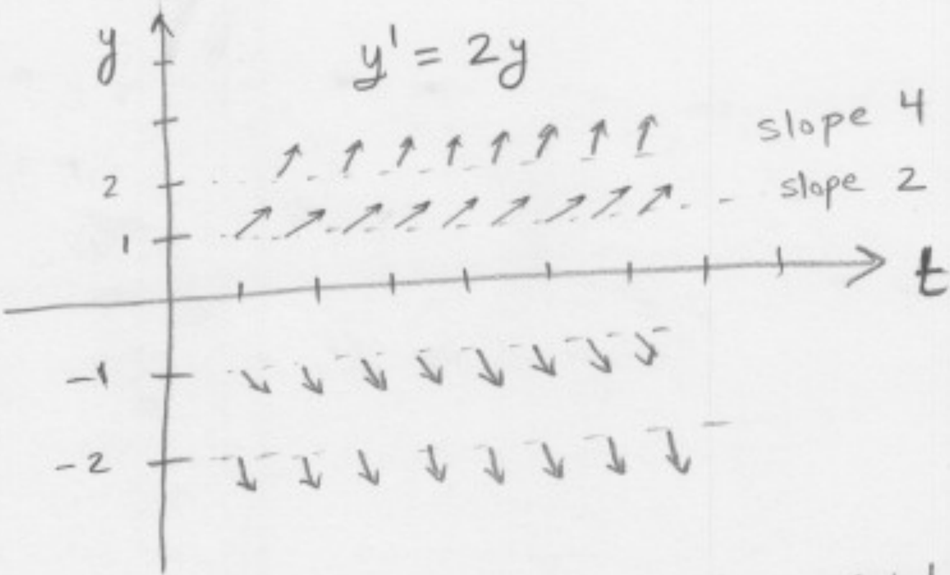
$$\lim_{t\to\infty} y(t) = -\infty, \quad \text{if } y(0) < 0$$

The solution $y(t) \equiv 0$ is called an equilibrium solution because it is a constant solution (horizontal curve).

In real applications, we may only know that solutions to a differential equation exist, but it might be impossible (or very hard) to have a formula for the solution. In these cases, we can always draw the direction field, using only the equation.

How to draw the direction field using only the equation? Given a t and a y, we plug them in the equation to find $y'(t)$, and we draw an arrow with slope $y'(t)$ at (t,y)

y	$y'(t)$
2	4
1	2
0	0
-1	-2
-2	4



Since $y' = 2y$ does not have t's on the right side, note that all the arrows have the same slope in a horizontal row of arrows (independent of the value of t).

However, if we had the equation, say, $y' = y^2 + e^t$, we would need to consider the t:

t	y	$y'(t)$
1	1	$y' = 1 + e^1$
2	0	$y' = 0 + e^2$
3	-1	$y' = 1 + e^3$

Matlab can plug many values of t, y in the equation and draw the arrows very fast.

Differential equation are divided in 2 major groups:

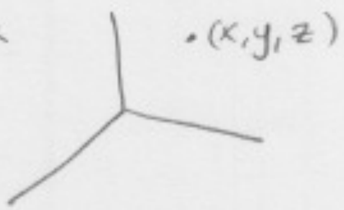
<u>Ordinary</u> differential equations: The word ordinary refers to the fact that the unknown is a function of 1 variable, $y(t)$.

<u>Ex</u>: $y' = y^2 + \sin t$

$$y''(t) - 2y'(t) + y(t) = e^t$$

<u>Partial</u> differential equations: The word partial refers to the fact that the unknown is a function of more than 1 variable.

Ex



$u(t, x, y, z)$ is the temperature of the room at time t, and at the location (x, y, z)

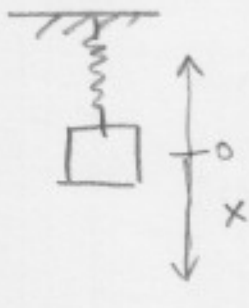
The heat equation is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad \text{or} \quad u_t = u_{xx} + u_{yy} + u_{zz}$$

To find a u that solves this equation is more advanced (see section 10.5 in your book).

We can use ordinary differential equations to model physical phenomena that can be described with only one variable

<u>Ex.</u>



$x(t)$ = displacement from equilibrium at time t

If you have a spring with a block, you pull down the block and let the spring move up and down in <u>one line</u>. The displacement can be measured with a finction $x(t)$ of one variable t.

In general, many fundamental differential equations that model fluid flow, electro-magnetism, heat, waves, etc, are <u>partial</u> since we live in more than one dimension.