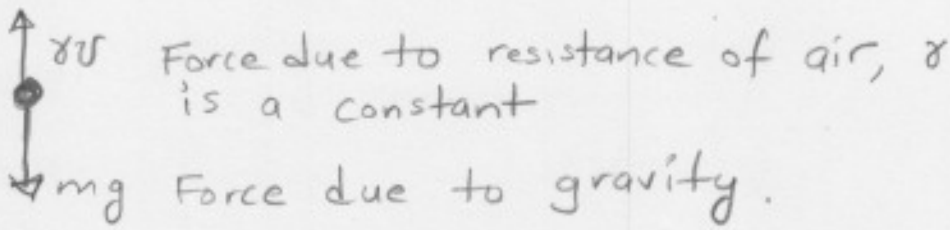
Section 1.2

Solutions of some differential equations

Two examples of modeling with differential equations.

<u>Example 1</u>: Free fall.

Formulate a differential equation describing motion of an object falling in the atmosphere near sea level.



Let $v(t)$ be the velocity of the object at time t. The physical law acting on the body is Newton's second law:

$$F = mg = m\frac{dv}{dt}.$$

F is the net force acting on the object, hence:

$$F = mg - \gamma v$$

$$\Rightarrow \qquad m\frac{dv}{dt} = mg - \gamma v$$

Taking $g = 9.8\ m/sec^2$, $m = 10\ kg$, $\gamma = 2\ kg/sec$ we obtain:

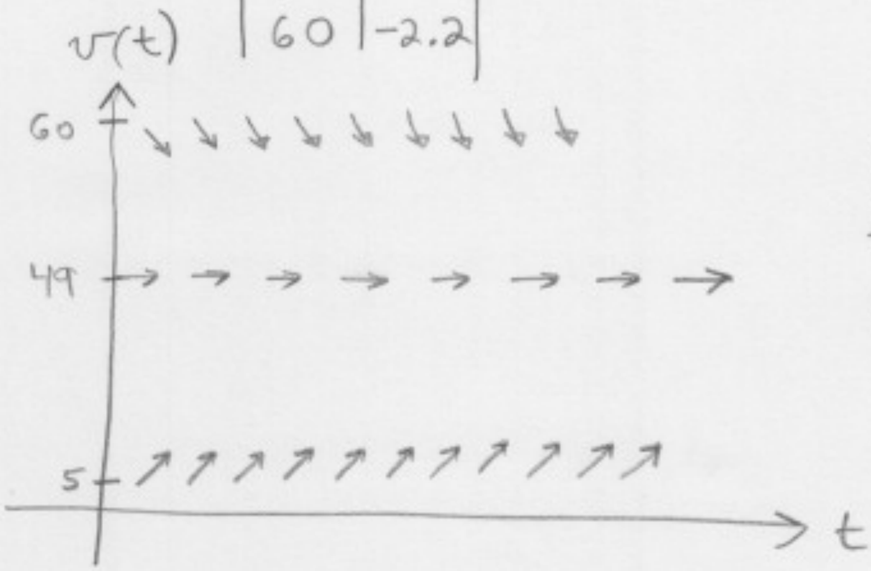
$$\frac{dv}{dt} = 9.8 - 0.2v$$

We have:

$$v' = 9.8 - 0.2\,v$$

We will later in these notes learn how to solve this differential equation. Even without solving the equation, we still can get a lot of information about the solutions from the direction field:

v	v'
0	9.8
5	8.8
10	7.8
49	0
50	-0.2
60	-2.2

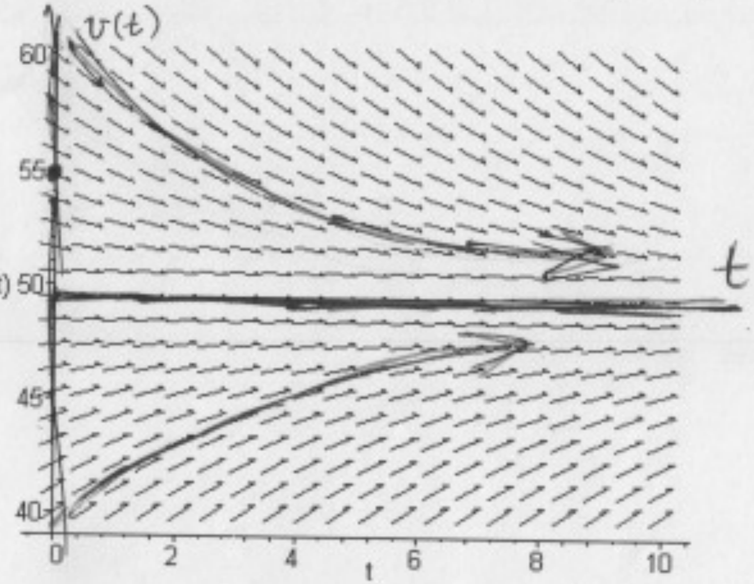


Use matlab to draw the direction field

$v' = 9.8 - 0.2v$

Example 1:
Direction Field & Equilibrium Solution (4 of 4)

- Arrows give tangent lines to solution curves, and indicate where soln is increasing & decreasing (and by how much).
- Horizontal solution curves are called **equilibrium solutions**.
- Use the graph below to solve for equilibrium solution, and then determine analytically by setting $v' = 0$.



Set $v' = 0$:

$\Leftrightarrow 9.8 - 0.2v = 0$

$\Leftrightarrow v = \dfrac{9.8}{0.2}$

$\Leftrightarrow v = 49$

This graph with all the arrows is called a "Direction field"

Analysis of direction field for:

$$v' = 9.8 - 0.2 \cdot$$

The only equilibrium solution (i.e. constant solution) is:

$$v(t) = 49, \quad \text{for every } t,$$

that is, if $v(0) = 49$, the the object continues to fall with velocity 49.

If the initial velocity is $v(0) > 49$, then the velocity starts to decrease and it approaches 49 as $t > \infty$.

If the initial velocity is $v(0) < 49$, then the velocity starts to increase and it approaches 49 as $t > \infty$.

Example 2 : Mice and Owls.

Consider a mouse population that reproduces at a rate proportional to the current population, with a rate constant equal to 0.5 mice/month (assuming no owls present). when owls are present, they eat the mice. Suppose that the owls eat 15 per day (average). Write a differential equation describing mouse population in the presence of owls. Assume that there are 30 days in a month.

Solution : We are looking for a function $p(t)$

We have:

$$\frac{dp}{dt} = 0.5\,p(t) - 450$$

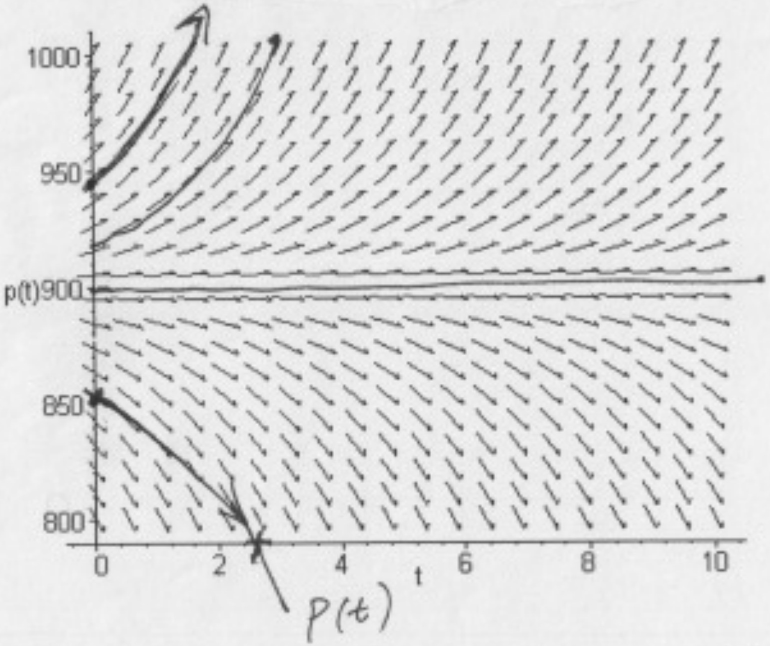
Analysis of the direction field: The only constant solution is $p(t) = 900$ for every t. This means that if the initial population of mice is $p(0) = 900$ then the population remains constant.

If $p(0) > 900$, then the population will explode to ∞.

If $p(0) < 900$, the eventually the owls will exterminate the population since at some tim t, $p(t)$ will be 0.

Example 5: Direction Field (2 of 2)

* Discuss solution curve behavior, and find equilibrium soln.



$p' = 0.5p - 450$.

$p(t) \equiv 900$

$0.5p - 450 = 0$

$p(t) = \dfrac{450}{\frac{1}{2}} = 900$

Horizontal solution curves are called <u>equilibrium solutions</u>

$p(t) = 900$ is the only horizontal <u>solution</u>.

Example 4:
Graphical Analysis for a Nonlinear Equation

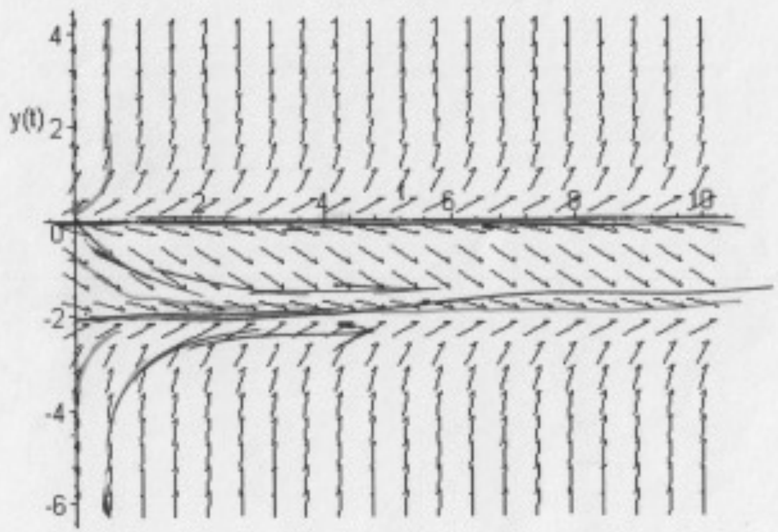
- Discuss solution behavior and dependence on the initial value $y(0)$ for the differential equation below, using the corresponding direction field.

$$y' = y(y+2) = y^2 + 2y.$$

equilibrium solutions.

$y(t) \equiv 0$

$y(t) = -2$



$y^2 + 2y = 0$

$y(y+2) = 0$

$y = 0 \quad y = -2$

The equations:

$$v' = -0.2v + 9.8$$
$$p' = 0.5p - 450$$

are both of the form:

$$\boxed{y'(t) = ay(t) - b} \quad (*)$$

We proceed to solve $(*)$:

$$y'(t) = a\left(y(t) - \frac{b}{a}\right)$$

$$\Rightarrow \quad \frac{y'(t)}{y(t) - \frac{b}{a}} = a \quad \rightarrow (1)$$

In (1) we see the equality of two functions of t, hence we integrate both sides in (1) with respect to t, and we obtain again equality up to a constant:

$$\int \frac{y'(t)}{y(t) - \frac{b}{a}}\, dt = \int a \, dt.$$

Let $v = y(t) - \frac{b}{a} \Rightarrow dv = y'(t)\, dt$.

We know that $\int \frac{dv}{v} = \int \frac{d}{dv}(\ln|v|) = \ln|v| + C$.

$$\ln\left|y(t) - \frac{b}{a}\right| = at + C.$$

Since $\ln x = y \iff e^y = x$ we obtain:

$$|y(t) - \tfrac{b}{a}| = e^{at+c} = e^c e^{at}$$

$$\Rightarrow \quad y(t) - \tfrac{b}{a} = \pm e^c e^{at}$$

Since c is arbitrary, we can rename this constant as C again

$$\Rightarrow \quad \boxed{y(t) = \frac{b}{a} + C e^{at}}$$

We have the family of solutions $y = \frac{b}{a} + Ce^{at}$, but if we want to find a particular solution curve, the initial condition $y(0)$ needs to be given to us:

Ex: Solve the initial value problem:

$$y' = ay - b, \quad y(0) = y_0.$$

The family of solutions is

$$y(t) = \frac{b}{a} + Ce^{at}$$

Plug $t = 0$:

$$y(0) = \frac{b}{a} + Ce^0 = \frac{b}{a} + C$$

Since $y(0) = y_0 \Rightarrow C = y_0 - \frac{b}{a}$

Hence, the particular solution is:

$$\boxed{y(t) = \frac{b}{a} + \left(y_0 - \frac{b}{a}\right)e^{at}}$$

<u>Remark</u>: The art of solving differential equations is to manipulate the equation in such a way that we can apply the fundamental theorem of calculus, that says that the integral of an exact derivative is the function itself:

<u>Fundamental theorem of Calculus (FTC)</u>:

<u>Part 1</u>: $\int_a^b f'(x) = f(b) - f(b)$

<u>Part 2</u>: If we form the function

$$h(t) = \int_0^t f(x)\,dx,$$

then $h'(t) = f(t)$

Indeed, for the equation $y' = ay - b$, we arrived at:

$$\frac{y'(t)}{y(t) - \frac{b}{a}} = a, \text{ which is the same as:}$$

$$\frac{d}{dt}\left(\ln\left|y(t) - \frac{b}{a}\right|\right) = a$$

and then we can integrate in both sides:

$$\int \frac{d}{dt}\left(\ln\left|y(t) - \frac{b}{a}\right|\right)dt = \int a\,dt$$, and we applied the FTC (page 15)

Remark: In practice, even though is not mathematically rigorous, we proceed as follows:

$$\frac{dy}{dt} = ay - b$$

$$\Rightarrow \quad dy = a\left(y - \frac{b}{a}\right) dt \, ,$$

We split the symbol, not really correct, but it works for practical computations"

$$\Rightarrow \quad \frac{dy}{y - \frac{b}{a}} = a\,dt$$

Integrate:

$$\int \frac{dy}{y - \frac{b}{a}} = \int a\,dt$$

$$\ln \left| y - \frac{b}{a} \right| = at$$

$$\Rightarrow \left| y - \frac{b}{a} \right| = e^{at+c}$$

$$y - \frac{b}{a} = C e^{at}$$

$$y(t) = \frac{b}{a} + C e^{at}.$$

Ex:

$$p' = 0.5p - 450$$

$$\frac{dp}{dt} = 0.5\,(p-900)$$

$$\int \frac{dp}{p-900} = \int 0.5\, dt$$

$$\ln |p-900| = 0.5t$$

$$|p-900| = e^{0.5t+c} = e^{c} e^{0.5t}$$

$$p - 900 = \pm e^{c} e^{0.5t}$$

$$\boxed{p(t) = 900 + c\, e^{0.5t}},$$

since c is arbitrary we can replace $\pm e^{c}$ with just c

Ex: If the initial population is $p(0) = 850$ mice, find the solution corresponding to this initial condition.

$$p(t) = 900 + c\, e^{0.5t}$$

$$\underset{850}{\underset{\parallel}{p(0)}} = 900 + c \implies c = -50$$

$$\implies p(t) = 900 - 50\, e^{0.5t}$$

From the direction field we see that, for some t_0, $p(t_0) = 0$. Hence the population will dissapear at $t = t_0$.