Section 1.3

Classification of differential equations

In the most general form, an ordinary differential equation can be writen as follows:

$$y^{(n)}(t) = f(t, y, y', y'', \dots, y^{(n-1)})$$

<u>Ex</u>:

$y'' + 3e^{y}y' - 2t = 0$ can be written in the form:

$$y'' = -3e^{y}y' + 2t = f(t, y, y')$$

$y'''' - ty'' + 1 = t^2$ can be written in the form:

$$y'''' = ty'' - 1 + t^2 = f(t, y'')$$

<u>Def</u>: The order of a differential equation is the order of the highest derivative that appears in the equation.

Ex:

$y' + 3y = 0$, first order

$y'' + 3y' - 2t = 0$, second order

$y'''' - y'' + 1 = e^t$, fourth order

We say that $\phi(t)$ is a solution to the ordinary differential equation:

$$y^{(n)}(t) = f(t, y, y', \dots, y^{(n-1)})$$

if when we plug $\phi(t)$ in the equation, the equation is true; that is:

$$\phi^{(n)}(t) = f(t, \phi, \phi', \dots, \phi^{(n-1)})$$

Ex: Verify the following solutions of the ODE:

$$y'' + y = 0$$

$$y_1 = \sin t, \quad y_2(t) = \cos t, \quad y_3(t) = 2\sin t$$

$y_1' = \cos t$, $y_1'' = -\sin t$, y_1 is a solution because:

$$y_1'' + y_1 = (-\sin t) + \sin t = 0, \text{ for every } t$$

$y_2' = -\sin t$, $y_2'' = -\cos t$, y_2 is a solution because:

$$y_2'' + y_2 = -\cos t + \cos t = 0, \text{ for every } t.$$

Actually, any function $\phi(t) = c_1 \sin t + c_2 \cos t$ solves the equation (check it), where c_1, c_2 can be any numbers, including 0.

Differential equations can be classified in 2 large groups: linear equations and non-linear equations.

In order to better understand these concepts, we recall from linear algebra:

Remark from Linear Algebra:

Let V be a vector space and $L: V \to V$ a function. We say that L is linear if for every v_1, v_2 in V and α a real number:

$$L(v_1+v_2) = L(v_1)+L(v_2)$$
$$L(\alpha v) = \alpha L(v), \text{ for every } v \text{ in } V.$$

Ex: $L: \mathbb{R} \to \mathbb{R}$

$L(x) = x^2$ is non-linear because

$$L(x_1+x_2) = (x_1+x_2)^2 \neq \underbrace{L(x_1)+L(x_2)}_{= x_1^2 + x_2^2}$$

$L(x) = e^x$ is non-linear because

$$L(x_1+x_2) = e^{x_1+x_2} \neq \underbrace{L(x_1)+L(x_2)}_{= e^{x_1}+e^{x_2}}$$

$L(x) = 5x$ is linear because

$$L(x_1+x_2) = 5(x_1+x_2) = 5x_1+5x_2 = L(x_1)+L(x_2)$$

<u>Def</u>: An ordinary differential equation

$$F(t, y, y', y'', y''', \dots, y^{(n)}) = 0 \quad (*)$$

is linear if F is linear in the variables

$$y, y', y'', y''', \dots, y^{(n)}$$

Therefore, the general ordinary differential equation $(*)$ is linear <u>if and only if</u> it is of the form:

$$\boxed{a_0(t) y^{(n)} + a_1(t) y^{(n-1)} + \dots + a_n(t) y = g(t) \,.} \quad (**)$$

<u>Remark</u>: Let us check that $(**)$ is linear in, say y'' (the same argument holds for the other variables, $y, y', y''', \dots, y^{(n)}$).

Let L be the map acting on functions as follows:

$$L(y) = a_{n-2}(t) y''$$

that is, if we input a function y to L, the output is the second derivative of y multiplied by the function of t, $a_{n-2}(t)$, as in $(**)$. L is linear because:

$$L(y_1 + y_2) = a_{n-2}(t)\,(y_1+y_2)'' = a_{n-2}(t)\,(y_1'' + y_2'')$$
$$= a_{n-2}(t) y_1'' + a_{n-2}(t) y_2'' = L(y_1) + L(y_2$$

Thus, in order to determine if an ODE is linear or non-linear we compare it with (**). It is linear <u>if and only if</u> it is of the form (**).

Ex. Determine whether the equations below are linear or non-linear.

① $y' + 3y = 0$ linear

② $y'' + 3e^{y}y' - 2t = 0$ or $y'' + 3e^{y}y' = 2t$, non-linear

③ $y'' + 3y' - 2t^2 = 0$ or $y'' + 3y' = 2t^2$, linear

④ $y'''' - t\,y'' + 1 = t^2$ or $y''' - ty'' = -1 + t^2$, linear

Clearly, ①, ③ and ④ are of the form (**) and hence they are linear.

Notice, in (**), that the coefficients $a_i(t)$, $i = 0, \dots, n$ are functions of t. They can be constants or zero in particular, but they can not contain y or its derivatives.

Thus, ② is non-linear because the coefficient in front of y' depends on y.