Section 2.1

Method of integrating factors

Consider the general form of a linear, first order ODE:

$$a_0(t)\, y' + a_1(t)\, y = \tilde{g}(t), \qquad a_0(t) > 0$$

We first put a 1 in front of y':

$$y' + \frac{a_1(t)}{a_0(t)}\, y = \frac{\tilde{g}(t)}{a_0(t)}$$

$$\text{Let } p(t) := \frac{a_1(t)}{a_0(t)}, \quad g(t) := \frac{\tilde{g}(t)}{a_0(t)}$$

Hence, the equation can be written as:

$$\boxed{y' + p(t)\, y = g(t)}$$

In order to produce an exact derivative on the left, we multiply the equation by a function $\mu(t) > 0$ which will be determined later.

$$\underbrace{\mu(t) y' + p(t) \mu(t)\, y}_{(A)} = \mu(t)\, g(t) \qquad (*)$$

We note that:

$$\frac{d}{dt}(\mu y) = \underbrace{\mu y' + y\mu'}_{(B)} \qquad (**)$$

We compare (A) and (B). Both have the term $\mu y'$ but the other term do not agree. (A) contains $p(t)\mu(t)y$ and (B) has $\mu' y$. Can we find $\mu(t)$ such that:

$$\mu'(t) = p(t)\mu(t) \qquad (***)$$

The answer is yes. With this μ, we have that (A) = (B) and therefore from (*) and (**) we obtain:

$$\frac{d}{dt}(\mu y) = \mu(t) g(t)$$

With such $\mu(t)$, we now have an exact derivative on the left part of the equation. We can integrate both sides with respect to t, and use the fundamental theorem of calculus to recover $y(t)$:

$$\cancel{\int \frac{d}{dt}} (\mu(t) y(t))\, dt = \int \mu(t)\, g(t)\, dt$$

$$\mu(t)\, y(t) = \int \mu(t) g(t)\, dt + C$$

Hence:

$$\boxed{y(t) = \frac{1}{\mu(t)} \int \mu(t)\, g(t)\, dt + \frac{c}{\mu(t)}} \quad (1)$$

where C is any number. We have found the family of solutions to the first order, linear ODE $y' + p(t)y(t) = g(t)$

How to find $\mu(t)$? From (***) we recall that $\mu(t)$ needs to solve:

$$\mu'(t) = \mu(t)\, p(t), \quad \mu(t) > 0$$

$$\Rightarrow \frac{\mu'(t)}{\mu(t)} = p(t)$$

$$\Rightarrow \int \frac{d}{dt}(\ln \mu(t)) = \int p(t)\, dt$$

$$\ln \mu(t) = \int p(t)\, dt + C \ ; \quad \text{choose } C = 0$$

$$\Rightarrow \boxed{\mu(t) = e^{\int p(t) dt}}$$

Remark: The general solution (1) requires to perform the integral $\int \mu(t) g(t)\, dt$. Hence, $g(t)$ must be continuous, $p(t)$ needs to be continuous too. You learn in real analysis that the Riemann integral exists for continuous functions. If $g(t)$ is continuous, but $\int \mu(t) g(t)\, dt$ is hard to compute in practice, the only way to compute it might be using a numerical method of integration.

<u>Ex</u>: Solve $y' + 2y = e^{t/2}$

$$P(t) = 2 \qquad \mu(t) = e^{\int 2\,dt} = e^{2t}$$

$$e^{2t} y' + 2e^{2t} y = e^{t/2} e^{2t}$$

$$\frac{d}{dt}\left(e^{2t} y\right) = e^{\frac{5t}{2}}$$

$$\int \frac{d}{dt}\left(e^{2t} y\right) = \int e^{\frac{5}{2}t}\,dt$$

$$e^{2t} y(t) = \frac{2}{5} e^{\frac{5}{2}t} + C$$

$$\Rightarrow \boxed{y(t) = \frac{2}{5} e^{\frac{t}{2}} + C e^{-2t}}$$

(See direction field at the end of the lecture)

<u>Ex</u>: Solve $y' + \frac{1}{5} y = 5 - t$

$$P(t) = \frac{1}{5}$$

$$\mu(t) = e^{\int \frac{1}{5}\,dt} = e^{\frac{t}{5}}$$

$$y' e^{t/5} + \frac{1}{5} e^{t/5} y = e^{\frac{t}{5}}(5-t)$$

$$\Rightarrow \frac{d}{dt}\left(y e^{t/5}\right) = e^{\frac{t}{5}}(5-t)$$

$$\int \frac{d}{dt}\left(y e^{t/5}\right) dt = \int e^{\frac{t}{5}}(5-t)\,dt$$

$$y(t)\, e^{t/5} = \int e^{\frac{t}{5}}(5-t)\,dt$$

$$\int e^{\frac{t}{5}}(5-t)\,dt = \int 5 e^{\frac{t}{5}}\,dt - \int e^{\frac{t}{5}} t\,dt$$

$$= 25 e^{\frac{t}{5}} - \int t e^{\frac{t}{5}}\,dt$$

Let $u = t$ $\quad dv = e^{\frac{t}{5}}$

$du = dt$ $\quad v = 5e^{\frac{t}{5}}$

$\int u\,dv = uv - \int v\,du$

$$= 25 e^{\frac{t}{5}} - 5t e^{\frac{t}{5}} + \int 5 e^{\frac{t}{5}}\,dt$$

$$= 25 e^{\frac{t}{5}} - 5t e^{\frac{t}{5}} + 25 e^{t/5}$$

$$= 50 e^{t/5} - 5t e^{t/5} + C$$

$$\therefore\ y(t)\, e^{t/5} = 50 e^{t/5} - 5t e^{t/5} + C$$

$$\boxed{y(t) = 50 - 5t + Ce^{-t/5} \quad C \text{ any number}}$$

Infinite family of solutions

Ex: Find the particular solution that passes through $(0, 50)$

$$\underset{\overset{\|}{50}}{y(0)} = 50 + C e^{0} = 50 + C$$

$$\therefore\ C = 0$$

This solution is the line $y(t) = 50 - 5t$

(see direction field at the end of lecture).

Ex: Solve

$$ty' - 2y = 5t^2$$

$$y' - \frac{2}{t}y = 5t, \quad t \neq 0$$

$$p(t) = -2/t$$

$$\mu(t) = e^{\int -\frac{2}{t}\,dt} = e^{-2\ln|t|} = e^{\ln|t|^{-2}} = \frac{1}{t^2}$$

We multiply both sides by $\frac{1}{t^2}$:

$$\frac{1}{t^2}y' - \frac{2}{t^3}y = \frac{5}{t}$$

$$\frac{d}{dt}\left(\frac{1}{t^2}y(t)\right) = \frac{5}{t}$$

$$\int \frac{d}{dt}\left(\frac{1}{t^2}y(t)\right)dt = \int \frac{5}{t}\,dt$$

$$\frac{1}{t^2}y(t) = 5\ln|t| + C$$

$$\boxed{\begin{array}{c} y(t) = 5t^2\ln|t| + Ct^2 \\ C \text{ any real number} \end{array}}$$

Ex: Solve the initial value problem:

$$\begin{cases} ty' - 2y = 5t^2 \\ y(1) = (2) \end{cases}$$

We have found the family of solutions:

$$y(t) = 5t^2 \ln|t| + ct^2$$

$$\Rightarrow \underset{\overset{\parallel}{2}}{y(1)} = 5\ln|1| + c(1) = c \quad \Rightarrow c = 2$$

Hence, the particular solution that passes through $(1,2)$ is:

$$y(t) = 5t^2 \ln|t| + 2t^2$$

See the attached graph with many solutions belonging to the family, corresponding to different values of the constant c. The solution that passes through $(1,2)$ is also remarked in the picture.

Example 1: General Solution (2 of 2)

* With $\mu(t) = e^{2t}$, we solve the original equation as follows:

$$y' + 2y = e^{t/2}$$

$$\mu(t)\frac{dy}{dt} + 2\mu(t)y = \mu(t)e^{t/2}$$

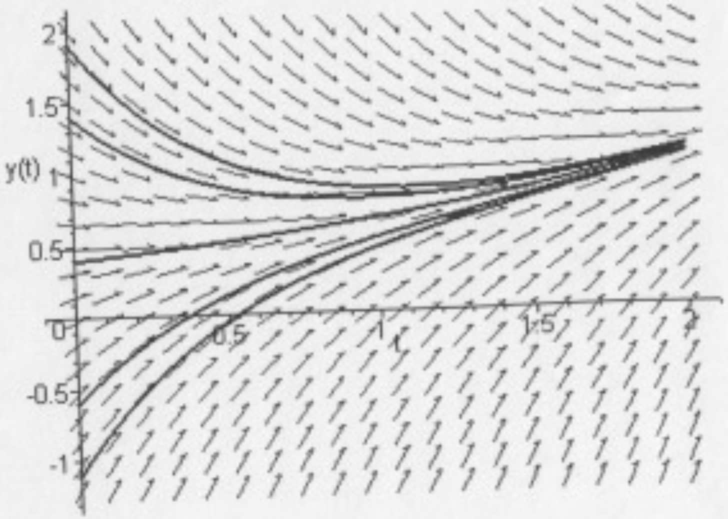
$$e^{2t}\frac{dy}{dt} + 2e^{2t}y = e^{5t/2}$$

$$\frac{d}{dt}\left[e^{2t}y\right] = e^{5t/2}$$

$$e^{2t}y = \frac{2}{5}e^{5t/2} + C$$

$$y = \frac{2}{5}e^{t/2} + Ce^{-2t}$$

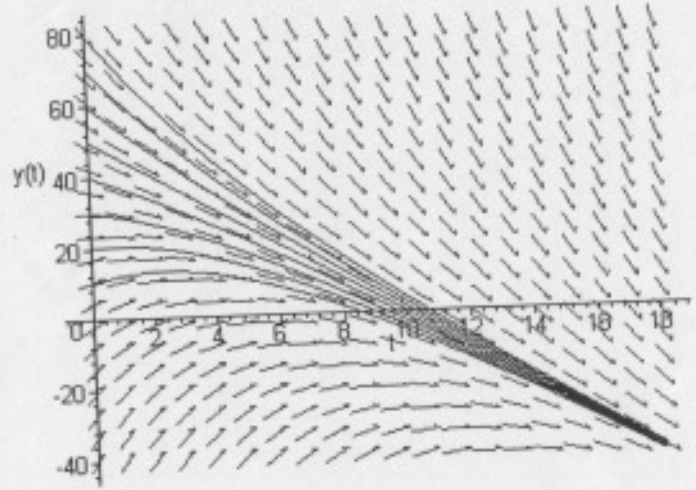
y(t) = 2/5 exp(t/2) + C exp(-2t)

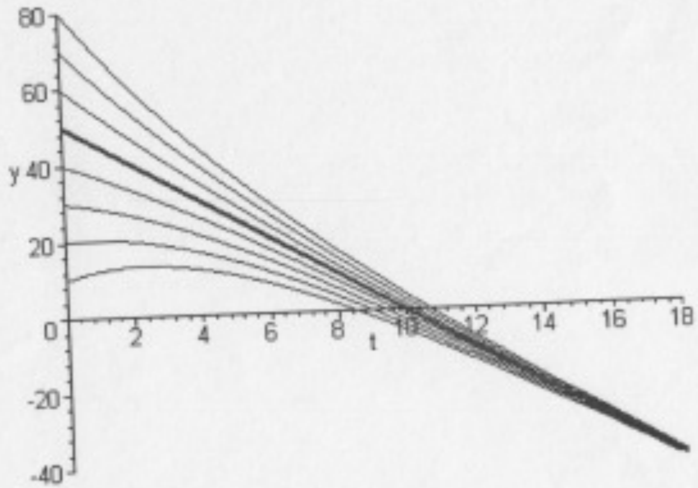


Example 2: Graphs of Solutions (2 of 2)

- The graph on left shows direction field along with several integral curves.
- The graph on right shows several solutions, and a particular solution (in red) whose graph contains the point (0,50).

$$y' = -\frac{1}{5}y + 5 - t \;\Rightarrow\; y = 50 - 5t + Ce^{-t/5}$$





Example 4: Graphs of Solution (3 of 3)

- The graphs below show several integral curves for the differential equation, and a particular solution whose graph contains the initial point (1,2).

IVP: $ty' - 2y = 5t^2, \quad y(1) = 2$

General Solution: $y = 5t^2 \ln|t| + Ct^2$

Particular Solution: $y = 5t^2 \ln|t| + 2t^2$