Section 2.2

Separable differential equations

Ex: Find a function $y(x)$ that solves the following ODE:

$$y'(x) = \frac{x^2+1}{y^2-1}$$

We can write this equation as:

$$(y^2-1)\, y'(x) = 1+x^2 .$$

We note that it is a first order non-linear equation, since the coefficient of $y'(x)$ depends on y.

We see that we can write the eq. as:

$$-(1+x^2) + (y^2-1)\, y'(x) = 0 ,$$

and that it is of the form

$$M(x) + N(y)\, y'(x) = 0 , \longrightarrow (1)$$

with $M(x) = -(1+x^2)$ and $N(y) = y^2-1$

We can solve an equation of the form (1) if we can find functions $H_1(x)$ and $H_2(y)$ such that:

$$H_1'(x) = M(x) \qquad \text{and} \qquad H_2'(y) = N(y)$$

Indeed, if this is the case we have:

$$H_1'(x) + H_2'(y)\, y'(x) = 0 \longrightarrow (2)$$

We now recall the chain rule from Calculus I: The derivative of the composition $f \circ g$ is:

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

Therefore, the second term in (2) is:

$$\frac{d}{dx} H_2(y(x)) = H_2'(y(x)) \cdot y'(x),$$

since y is a function of x.

Hence, (2) can be rewritten as:

$$\frac{d}{dx} H_1(x) + \frac{d}{dx} H_2(y(x)) = 0$$

or

$$\frac{d}{dx} \left(H_1(x) + H_2(y) \right) = 0$$

We were able to write the left side of the equation as an exact derivative. We now integrate both sides of the equation:

$$\int \frac{d}{dx} \left(H_1(x) + H_2(y) \right) = \int 0$$

or $\boxed{H_1(x) + H_2(y) = C}$

The solution $y(x)$ of eq. (1) is given implicitly by:

$H_1(x) + H_2(y) = C$, where C is any real number.

Going back to our example:

$$-(1+x^2) + (y^2-1)\,y'(x) = 0$$

we have:

$$M(x) = -(1+x^2) \qquad N(y) = y^2 - 1$$

$$H_1(x) = -x - \frac{x^3}{3} \qquad H_2(y) = \frac{y^3}{3} - y$$

The solution is

$$-x - \frac{x^3}{3} + \frac{y^3}{3} - y = c, \text{ or}$$

$$\boxed{y^3 - 3y - x^3 - 3x = C}$$

The direction field is at the end of the lecture.

Ex: Find $y(x)$ that solves the equation:

$$y'(x) = \frac{3x^2+4x+2}{2(y-1)}$$

We have:

$$-(3x^2+4x+2) + 2(y-1)\,y'(x) = 0$$

This is a first order non-linear equation, since the coefficient of y' depends on y. Following our method we have:

$$M(x) = -(3x^2+4x+2), \qquad N(y) = 2(y-1)$$

$$H_1(x) = -x^3-2x^2-2x \qquad H_2(y) = y^2-2y$$

The solution is:

$$H_1(x) + H_2(y) = C, \quad \text{or}$$

$$-x^3-2x^2-2x+y^2-2y = C$$

Hence, the solution is given implicitly by the equation:

$$y^2-2y-x^3-2x^2-2x = C$$

In this case, we can solve y as a function of x as follows:

$$y^2-2y-x^3-2x^2-2x-c=0$$

$$y=\frac{2\pm\sqrt{4-4(-x^3-2x^2-2x-c)}}{2}$$

$$y=\frac{2\pm 2\sqrt{1+x^3+2x^2+2x+c}}{2}$$

$$y=1\pm\sqrt{x^3+2x^2+2x+c}$$

Ex: Solve the initial value problem (IVP):

$$\begin{cases} y'(x)=\dfrac{3x^2+4x+2}{2(y-1)} \\ y(0)=3 \end{cases}$$

The general solution is

$$y^2-2y-x^3-2x^2-2x=c$$

We plug $y(0)=3$:

$$3^2-2(3)-0-0-0=c \Rightarrow c=9-6=3$$

$$\Rightarrow \boxed{y^2-2y-x^3-2x^2-2x=3}$$

We see from the direction field (at the end of this lecture) that $(0,3)$ belongs to the upper branch of solution. Thus we can use the explicit solution:

$$y(x)=1+\sqrt{x^3+2x^2+2x+c}$$

and plug $(0,3)$ there:

$$\underset{\overset{\shortparallel}{3}}{y(0)} = 1 + \sqrt{0 + C}$$

$$\Rightarrow \sqrt{C} = 2 \quad \Rightarrow \quad C = 4$$

Hence, the particular solution that passes through $(0,3)$ is:

$$y(x) = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$

or implicitly by $\quad y^2 - 2y - x^3 - 2x^2 - 2x = 3.$ ∎

Ex: Find the domain of the solution $y(x)$ that passes through $(0,3)$:

$$\begin{aligned} y(x) &= 1 + \sqrt{x^3 + 2x^2 + 2x + 4} \\ &= 1 + \sqrt{x^2(x+2) + 2(x+2)} \\ &= 1 + \sqrt{(x+2)(x^2+2)} \end{aligned}$$

We note that $y(x)$ is not defined for $x < -2$, so the domain is $[2, \infty)$, as you can see in the picture.

Ex: Solve the initial value problem:

$$y'(x) = \frac{y \cos x}{1+3y^3}, \quad y(0)=1$$

We have

$$\frac{1+3y^3}{y} y'(x) = \cos x,$$

hence this is a first order non-linear equation. We write

$$-\cos x + \frac{1+3y^3}{y} y'(x) = 0$$

$$M(x) = -\cos x \qquad N(y) = \frac{1+3y^3}{y} = \frac{1}{y} + 3y^2$$

$$\Rightarrow \quad H_1(x) = -\sin x \qquad H_2(y) = \ln|y| + y^3$$

The solution is:

$$H_1(x) + H_2(y) = C, \quad \text{or}$$

$$-\sin x + y^3 + \ln|y| = C$$

Using the initial condition we obtain:

$$-\sin 0 + 1 + \ln|1| = C$$

$$\Rightarrow \quad C = 1$$

$$\Rightarrow \quad \boxed{-\sin x + y^3 + \ln y = 1}$$

See the graph of solutions (at the end of lecture) and note that the solution that passes through $(0,1)$ is above $y=0$, so we can remove the absolute value in $\ln|y|$.

$\boxed{\text{Remark}}$: In practice, whenever we have:

$$M(x) + N(y)\, y'(x) = 0$$

we can separate the variables as:

$$M(x) + N(y)\frac{dy}{dx} = 0$$

$$N(y)\frac{dy}{dx} = -M(x)$$

$$N(y)\, dy = -M(x)\, dx \, ,$$

and integrate both sides as in Calculus 1. It is not really mathematically correct to split $y'(x)$, but for practical purpuses we can do it, since we end up with the same answer (but we must remember the rigorous explanation in page 36).

<u>Ex</u>: $y'(x) = \dfrac{y \cos x}{1+3y^3} \Rightarrow \dfrac{dy}{dx} = \dfrac{y\cos x}{1+3y^3} \Rightarrow$

$\dfrac{1+3y^3}{y}\, dy = \cos x \, dx \Rightarrow \int\left(\frac{1}{y} + 3y^2\right) dy = \int \cos x \, dx$

same answer $\Rightarrow \boxed{\ln|y| + y^3 = \sin x + C}$

Ex: Consider the equation

$$\frac{dy}{dt} = 2ty + t$$

This is a first order linear equation, it is also a separable equation:

$$\boxed{y' - 2ty = t}$$

or

$$y' = t(2y+1)$$

$$\Rightarrow \quad \boxed{\frac{dy}{2y+1} = t\,dt}$$

Find the general solution using the two methods we have so far: method of multiplying factors and method of separation of variables. Show that you get the same answer.

Homogeneous equations.

Ex: Find $y(x)$ that solves the differential equation:

$$y'(x) = \frac{x^2 + xy + y^2}{x^2}$$

This is a first order non-linear equation, since:

$$x^2 y'(x) - xy = x^2 + y^2$$

is not of the form $a_0(x)y' + a_1(x)y = g(x)$,

Note that it is not separable:

$$x^2 \frac{dy}{dx} = x^2 + xy + y^2$$

$$\frac{dy}{x^2 + xy + y^2} = \frac{dx}{x^2}, \quad \text{not separable.}$$

We can make a change of variables

Let $v = \frac{y}{x} \Rightarrow y = xv$

$$\Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v \cdot 1$$

$$\Rightarrow \quad x\frac{dv}{dx} + \cancel{v} = 1 + \left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2 = 1 + \cancel{v} + v^2$$

$$\Rightarrow \quad x\frac{dv}{dx} = 1 + v^2$$

Hence

$$\int \frac{dv}{1+v^2} = \int \frac{dx}{x}$$

$$\tan^{-1} v = \ln|x| + C$$

$$\tan^{-1} \frac{y}{x} = \ln|x| + C$$

$$\Rightarrow \quad \frac{y}{x} = \tan(\ln|x| + c)$$

$$\Rightarrow \quad \boxed{y = x \tan(\ln|x| + C)}$$

Note: $\displaystyle\int \frac{dv}{1+v^2} = \int \frac{\sec^2\theta \, d\theta}{1+\tan^2\theta} = \int \frac{\sec^2\theta}{\sec^2\theta} d\theta$

let $v = \tan\theta$ $\qquad = \displaystyle\int d\theta$

$dv = \sec^2\theta \, d\theta$ $\qquad = \theta = \tan^{-1} v.$

An homogeneous equation is the one that contains a term $\frac{y}{x}$ and/or $\frac{x}{y}$ and it can be solved using the change of variables

$$v = \frac{y}{x}.$$

Note: The relation between y' and v' is obtained from $y(x) = x\,v(x)$, by differentiating this relation.

<u>Ex</u>: Find $y(x)$ solution to:-

$$y'(x) = \frac{x^2 + 3xy + y^2}{x^2}$$

This is a first order non-linear equation.

$$\Rightarrow \quad y'(x) = 1 + 3\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right)^2,$$

which is a homogeneous equation.

Let $\quad v = \frac{y}{x}$

$$y = xv \quad \Rightarrow \quad y' = xv' + v$$

We substitute ;

$$xv' + v = 1 + 3v + v^2$$

$$xv' = 1 + 2v + v^2 = (1+v)^2$$

$$\Rightarrow \quad \frac{dv}{(1+v)^2} = \frac{dx}{x}$$

$$\int (1+v)^{-2} dv = \int \frac{dx}{x}$$

$$\frac{(1+v)^{-1}}{-1} = \ln|x| + C$$

$$\Rightarrow \quad \frac{-1}{1+v} = \ln|x| + C$$

$$\Rightarrow \quad \frac{-1}{1 + \frac{y}{x}} = \ln|x| + C, \quad \text{or} \quad \boxed{\frac{-x}{x+y} = \ln|x| + C}$$

where C is any real number

Lesson 4 (Sec. 2.2).

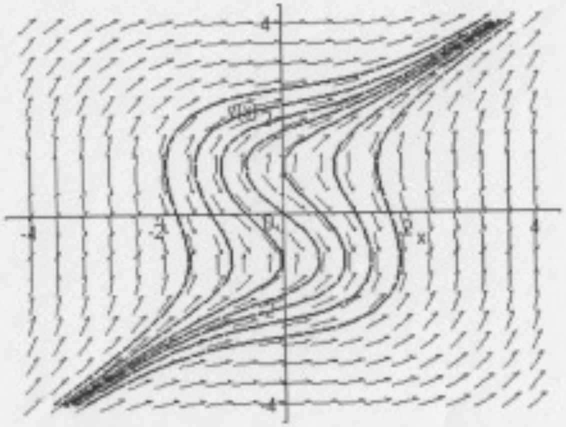
Example 1: Solving a Separable Equation

- Solve the following first order nonlinear equation:

$$\frac{dy}{dx} = \frac{x^2+1}{y^2-1}$$

- Separating variables, and using calculus, we obtain

$$(y^2-1)dy = (x^2+1)dx$$
$$\int (y^2-1)dy = \int (x^2+1)dx$$
$$\frac{1}{3}y^3 - y = \frac{1}{3}x^3 + x + C$$
$$y^3 - 3y = x^3 + 3x + C$$



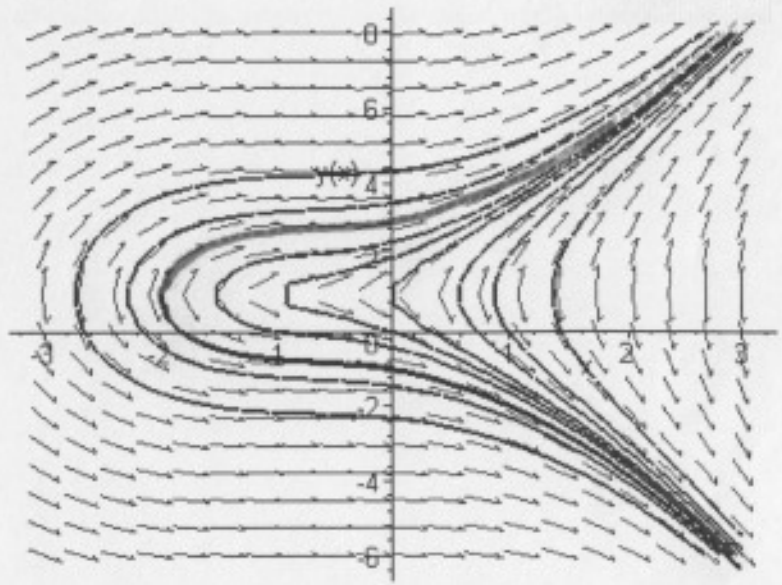
- The equation above defines the solution y implicitly. A graph showing the direction field and implicit plots of several integral curves for the differential equation is given above.

Example 2: Initial Condition $y(0) = 3$ (3 of 4)

- Note that if initial condition is $y(0) = 3$, then we choose the positive sign, instead of negative sign, on square root term:

$$y = 1 + \sqrt{x^3 + 2x^2 + 2x + 4}$$

$$y = 1 - \sqrt{x^3 + 2x^2 + 2x + 4}$$



Example 3: Graph of Solutions (2 of 2)

- Thus

$$y' = \frac{y\cos x}{1+3y^3}, \quad y(0)=1 \Rightarrow \ln y + y^3 = \sin x + 1$$

- The graph of this solution (black), along with the graphs of the direction field and several integral curves ([redacted]) for this differential equation, is given below.