Section 2.3

Modeling with first order differential equations.

Ex: Salt Solution

At time $t=0$, a tank contains Q_0 lb of salt disolved in 100 gal of water. Assume that water containing $1/4$ lb of salt / gal is entering the tank at rate of r gal/min, and leaves at the same rate.

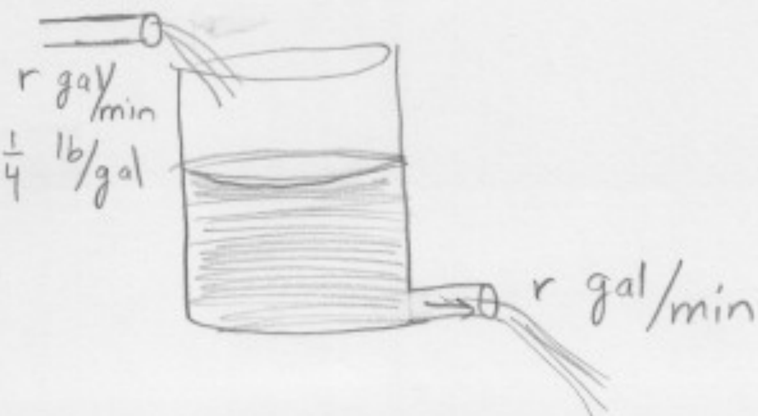
(a) Set up IVP that describes this salt solution flow process.

(b) Find amount of salt $Q(t)$ in tank at any given time t

(c) Find limiting amount Q_L of salt in the tank after a very long time

(d) If $r=3$ and $Q_0 = 2Q_L$, find the time T after which salt is within 2% of Q_L

(e) Find flow rate r required if T is not to exceed 45 min.



Solution:

The tank always contains 100 gal of water since the water enters tank at rate r gal/min, and leaves at the same rate.

The physical principle that governs this problem is conservation of mass: salt is neither created or destroyed in tank. Hence, the rate of change in salt is the difference of the rate in minus the rate out.

Let $Q(t)$ be the amount of salt at any given time t.. Hence, conservation of mass says:

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}.$$

Note: The distribution of salt in tank is uniform (stirred).

$$\text{Rate in} = \underbrace{\left(\tfrac{1}{4}\ \text{lb salt}/\text{gal}\right)}_{\text{concentration}}\underbrace{\left(r\ \text{gal}/\text{min}\right)}_{\text{velocity}} = \frac{r}{4}\ \text{lb}/\text{min}$$

Rate out: There is $Q(t)$ lbs of salt disolved in 100 gal of water at any given time t, hence the concentration of salt in the tank at any time t is:

$$\frac{Q(t)}{100}\ \text{lb}/\text{gal}$$

$\Rightarrow$

$$\text{Rate out} = \Big(\underbrace{\frac{Q(t)}{100}\ \text{lb/gal}}_{\text{concentration}}\Big)\Big(\underbrace{r\ \text{gal/min}}_{\text{velocity}}\Big)$$

$\Rightarrow$

$$\frac{dQ}{dt} = \frac{r}{4} - \frac{rQ}{100}$$

Our IVP is:

$$\begin{cases} \dfrac{dQ}{dt} + \dfrac{r}{100}Q = \dfrac{r}{4} \\ Q(0) = Q_0 \end{cases}$$

, this solves part (a) of the problem

This is a first order linear equation, we can solve it using the method of integrating factors:

$$\mu(t) = e^{\int \frac{r}{100} dt} = e^{\frac{r}{100}t}$$

$$\Rightarrow \int \frac{d}{dt}\left(e^{\frac{r}{100}t} Q(t)\right) = \int \frac{r}{4} e^{\frac{r}{100}t}$$

$$e^{\frac{r}{100}t} Q(t) = 25\, e^{\frac{r}{100}t} + C$$

$$\Rightarrow \quad Q(t) = 25 + C e^{-\frac{r}{100}t}$$

$$Q(0) = 25 + C = Q_0 \Rightarrow C = Q_0 - 25$$

Hence, solution to IVP is:

$$\boxed{Q(t) = 25 + (Q_0 - 25)\, e^{-\frac{r}{100}t}}$$

This is the solution to part (b).

(c) After a very long time:

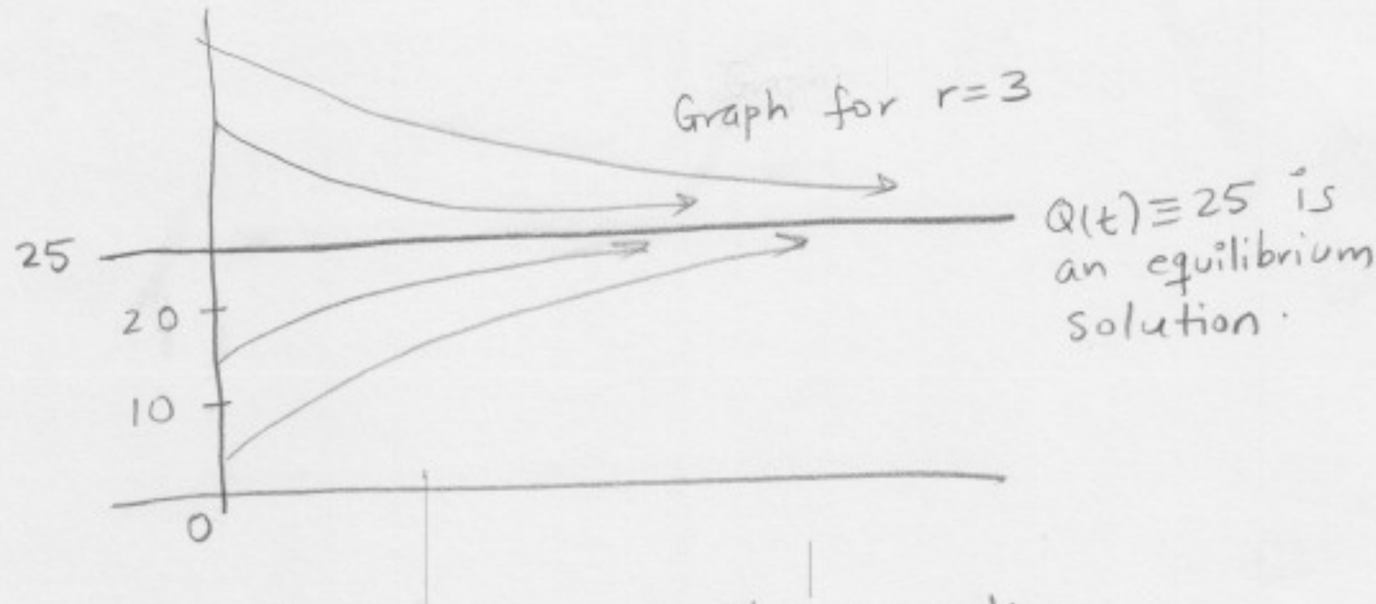
$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty}\left(25 + (Q_0 - 25)e^{\frac{-rt}{100}}\right)$$

$$= 25$$

Hence $Q_L = 25$.

Notice that $Q(t) \equiv 25$ is an equilibrium (or constant) solution to the equation.

This result makes sense, since over time the incoming salt solution will replace the original salt solution in tank. Since incoming solution contains 0.25 lb salt/gal, and the tank contains 100 gal, eventually the tank will contain 25 lb of salt.



Any other solution to the equation approaches the equilibrium solution as $t \to \infty$.

(d) If $r=3$ and $Q_0 = 2Q_L = 2(25) = 50$ the solution is:

$$Q(t) = 25 + (Q_0 - 25)\, e^{\frac{-rt}{100}}$$

$$= 25 + 25\, e^{-0.03t}$$

Next, 2% of 25 lb is 0.5 lb, and thus we solve:

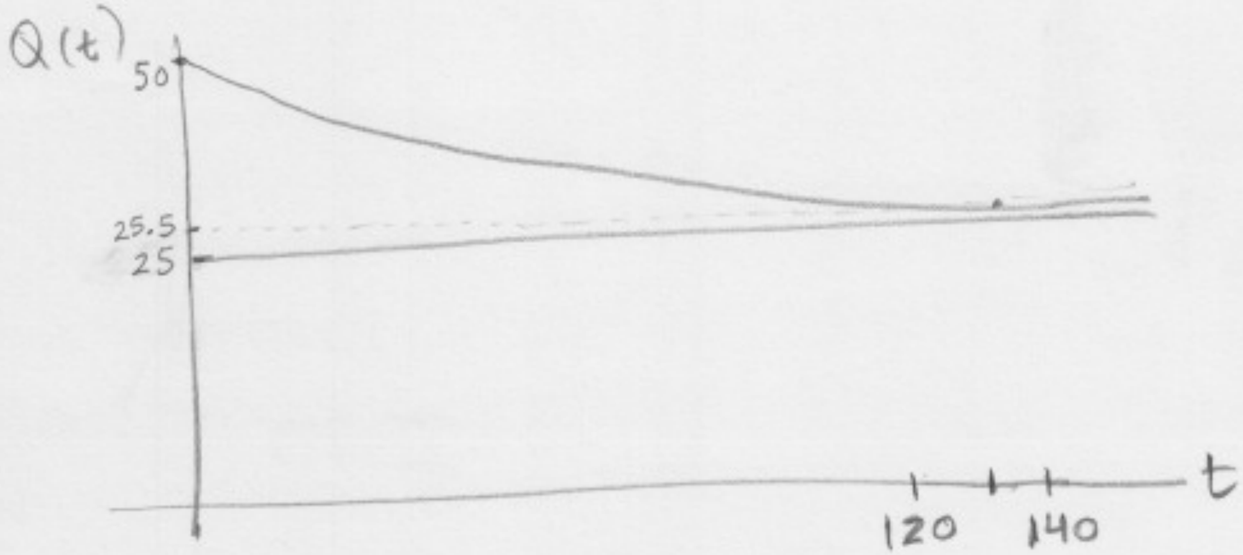
$$25.5 = 25 + 25\, e^{-0.03T}$$

$$0.5 = 25\, e^{-0.03T}$$

$$0.02 = e^{-0.03T}$$

$$\ln(0.02) = -0.03T$$

$$\Rightarrow \quad T = \frac{\ln(0.02)}{-0.03} \cong 130.4 \text{ min}$$



(e) To find the flow rate r required if T is not to exceed 45 minutes, recall from part (d) that $Q_0 = 2Q_L = 50$ lb.

We have:

$$Q(t) = 25 + 25\, e^{-\frac{r}{100}t}$$

We want $Q(45) = 25.5$

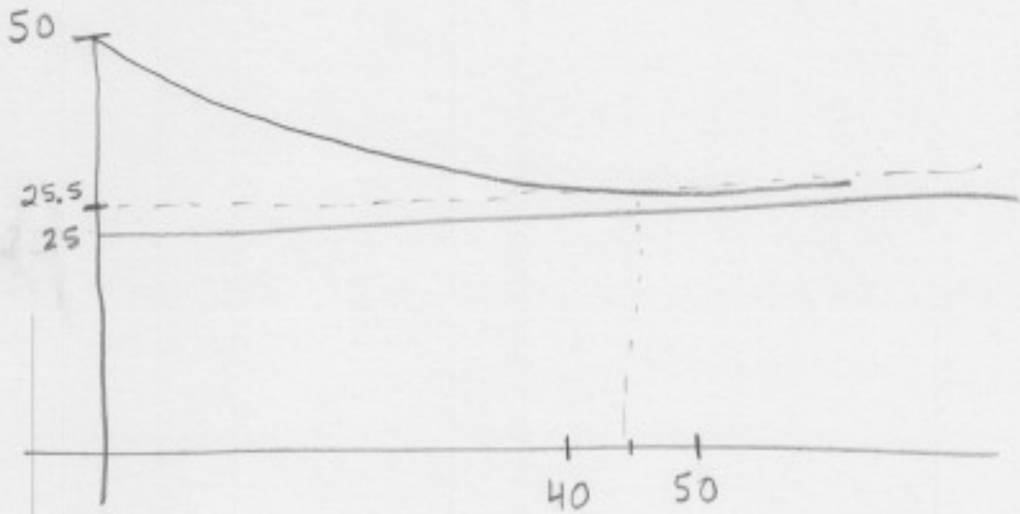
$$\Rightarrow \quad 25.5 = 25 + 25\, e^{-\frac{45}{100}r}$$

$$0.5 = 25\, e^{-0.45r}$$

$$0.02 = e^{-0.45r}$$

$$\Rightarrow \ln(0.02) = -0.45r$$

$$\Rightarrow \quad r = \frac{\ln(0.02)}{-0.45} \cong 8.69 \text{ gal/min}$$



Ex: A tank originally contains 100 gal of fresh water. Then water containing $\frac{1}{2}$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min the process is stopped, and fresh water is poured into the tank at a rate of 3 gal/min, with the mixture leaving again at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

Sol.: We split the problem in two parts. We model the first 10 min, and the we proceed to model the next 10 min.

For the first 10 min we have:

$Q(t)$ = amount of salt at time t
$Q(0) = 0$ (fresh water).

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$
$$= \left(\frac{1}{2}\right)(2) - \frac{Q(t)}{100}(2)$$
$$= 1 - \frac{Q(t)}{50}$$

We have the IVP:

$$\begin{cases} \dfrac{dQ}{dt} + \dfrac{1}{50}Q = 1 \\ Q(0) = 0 \end{cases}$$

$$\mu(t) = e^{\int \frac{1}{50} dt} = e^{\frac{t}{50}}$$

$$\Rightarrow \int \frac{d}{dt}\left(Q(t)\, e^{\frac{t}{50}}\right) = \int e^{\frac{t}{50}}$$

$$Q(t)\, e^{\frac{t}{50}} = 50\, e^{\frac{t}{50}} + C$$

$$Q(t) = 50 + C e^{-t/50}$$

$$Q(0) = 50 + C = 0 \Rightarrow C = -50$$

$$\Rightarrow Q(t) = 50 - 50\, e^{-t/50}$$

After 10 min:

$$Q(10) = 50 - 50\, e^{-10/50} \cong 9.063 \text{ lbs of salt.}$$

We can now model the next 10 minutes:

Let $P(t)$ = amount of salt

$$\frac{dP}{dt} = \text{rate in} - \text{rate out}$$

$$= \underset{\substack{\uparrow \\ \text{fresh} \\ \text{water}}}{(0)}(3) - \frac{P(t)}{100}(3) = \frac{-3}{100} P$$

Hence, the IVP is:

$$\begin{cases} \dfrac{dP}{dt} = \dfrac{-3}{100} P \\ P(0) = Q(10) = 9.063 \end{cases}$$

$$\Rightarrow \quad \frac{dP}{dt} = \frac{-3}{100} P$$

$$\int \frac{dP}{P} = \int -\frac{3}{100}\, dt$$

$$\ln |P| = \frac{-3}{100} t$$

$$\Rightarrow \quad P(t) = \pm e^{c} e^{-\frac{3}{100} t}$$

$$P(t) = C e^{-\frac{3}{100} t}$$

$$P(0) = C = 9.063$$

$\Rightarrow \quad P(t) = 9.063\, e^{-\frac{3}{100} t}$, this function gives us the amount of salt for the next 10 min.

$$\Rightarrow \quad P(10) = 9.063\, e^{-0.03(10)} \simeq 6.71 \text{ lbs.}$$

Ex: A tank with capacity of 500 gal of water originally contains 200 gal of water with 100 lb of salt in the solution. Water containing 1 lb of salt per gallon is entering at a rate of 3 gal/min and the mixture is allowed to flow out of the tank at a rate of 2 gal/min.

Let $Q(t)$ lb be the amount of salt in the tank, and $V(t)$ is the volume, in gallons, of water in the tank.

(a) Find the amount of salt in the tank at any time prior to the instant when the solution begins to overflow.

(b) Find the concentration (in lbs per gallon) of salt in the tank when it is on the point of overflowing.

Solution: $Q(0) = 100$

$$\frac{dQ}{dt} = \text{rate in} - \text{rate out}$$

$$= \left(1 \ \tfrac{\text{lb}}{\text{gal}}\right)\left(3 \ \text{gal}/\text{min}\right) - \left(\frac{Q(t)}{V(t)} \ \frac{\text{lb}}{\text{gal}}\right)\left(2 \ \frac{\text{gal}}{\text{min}}\right)$$

$$= 3 - 2\,\frac{Q(t)}{V(t)}$$

Note that $V(t) = 200 + t(3-2) = 200 + t$

$$\Rightarrow \frac{dQ}{dt} = 3 - \frac{2}{200+t} Q(t)$$

The IVP is:

$$\begin{cases} \frac{dQ}{dt} + \frac{2}{200+t} Q = 3 \\ Q(0) = 100 \end{cases}$$

This is a first order linear equation.

$$\mu(t) = e^{\int \frac{2}{200+t} dt} = e^{2 \ln |200+t|} = e^{\ln (200+t)^2}$$

$$= (200+t)^2$$

$$\int \frac{d}{dt} \left(Q(t) \, (200+t)^2 \right) = \int 3 \, (200+t)^2$$

$$Q(t) \, (200+t)^2 = (200+t)^3 + C$$

$$\Rightarrow Q(t) = (200+t) + C \, (200+t)^{-2}$$

$$\underset{\substack{\| \\ 100}}{Q(0)} = 200 + \frac{C}{200^2}$$

$$\Rightarrow \frac{C}{200^2} = -100 \qquad C = -100 \, (200)^2$$

$$\Rightarrow \boxed{Q(t) = 200 + t - \frac{100 \, (200)^2}{(200+t)^2}, \quad t < 300}$$

This is (a)

Note that $V(300) = 200 + 300 = 500$, and we arrive at maximal capacity, hence the above solution is valid for $t < 300$.

(b) $V(t) = 200 + t$

$$200 + t = 500$$

$$t = 300$$

Concentration at time $t = 300$ is:

$$\frac{Q(300)\ \text{lb}}{500\ \text{gal}} = \frac{500 - \frac{100(200)^2}{(500)^2}}{500}\ \text{lb/gal}.$$