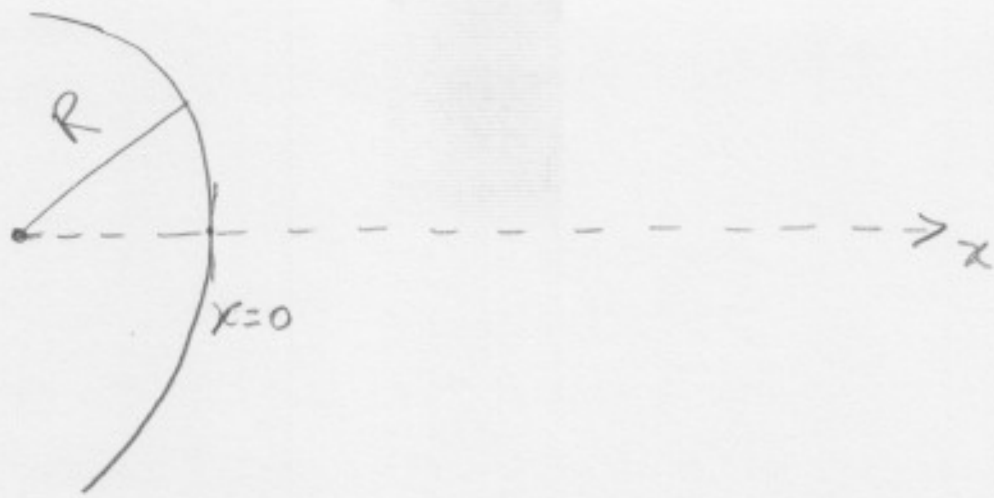
Section 2.3
Modeling with first order differential equations (continuation)

<u>Ex</u>: A body of constant mass is projected away from the earth in a direction perpendicular to the earth's surface with an initial velocity v_0.
Assuming that there is no air resistance, but taking in to account the variation of the earth's gravitational field with distance:

(a) Find an expression for the velocity during the motion

(b) Find the initial velocity v_0 that is required to lift the body to a given maximum altitude α above the surface of the earth

(c) Find the least initial velocity for which the body will not return to the earth (this is called the escape velocity)



Gravitational force acting on the body:

$$w(x) = \frac{-K}{(x+R)^2}$$

$$w(0) = -mg$$

$$\Rightarrow \quad -mg = \frac{-K}{R^2} \quad \Rightarrow \quad \boxed{K = mgR^2}$$

$$\Rightarrow \quad \boxed{w(x) = \frac{-mgR^2}{(x+R)^2}}$$

Since there are no other forces acting on the body, the equation of motion is:

$$F = ma$$

$$(1) \begin{cases} \dfrac{-mgR^2}{(x+R)^2} = m\dfrac{dv}{dt} \\ v(0) = v_0 \end{cases}$$

We choose x as the independent variable. Thus, we look for $v(x)$ instead of $v(t)$. We use the chain rule to accomplish this:

$$\frac{dv}{dt} = \frac{d}{dt}\left(v(x(t))\right)$$

$$= \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx} \quad ; \quad \text{since } v = \frac{dx}{dt}.$$

$$\Rightarrow \boxed{\frac{dv}{dt} = v\frac{dv}{dx}}$$

Hence, (1) becomes:

$$(2)\begin{cases} \dfrac{-mgR^2}{(x+R)^2} = m v \dfrac{dv}{dx} \\ v(0) = v_0. \end{cases}$$

We solve (2) using the method of separation of variables.

$$\int -gR^2(x+R)^{-2}\,dx = \int v\,dv$$

$$-gR^2\frac{(x+R)^{-1}}{-1} = \frac{v^2}{2} + C$$

$$\frac{gR^2}{x+R} = \frac{v^2}{2} + C$$

$\Rightarrow$ We impose $v(0) = v_0$

$$\frac{gR^2}{R} = \frac{v_0^2}{2} + C$$

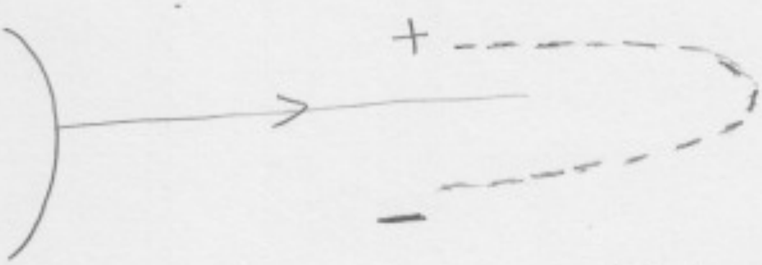
$$\Rightarrow C = gR - \frac{v_0^2}{2}$$

$$\Rightarrow \frac{v^2}{2} = \frac{gR^2}{x+R} + \frac{v_0^2}{2} - gR$$

$$v^2 = \frac{2gR^2}{x+R} + v_0^2 - 2gR$$

$$\Rightarrow \boxed{v(x) = \pm \sqrt{\frac{2gR^2}{R+x} + v_0^2 - 2gR}}$$

This is (a)



To compute the maximum altitute α that the body reaches we find α such that $v(\alpha) = 0$.

$$0 = \sqrt{v_0^2 - 2gR + \frac{2gR^2}{R+\alpha}}$$

$$\frac{2gR^2}{R+\alpha} = 2gR - v_0^2$$

$$R+\alpha = \frac{2gR^2}{2gR-v_0^2}$$

$$\alpha = \frac{2gR^2}{2gR-v_0^2} - R$$

$$\alpha = \frac{\cancel{2gR^2} - \cancel{2gR^2} + Rv_0^2}{2gR-v_0^2}$$

$$\alpha(v_0) = \frac{Rv_0^2}{2gR-v_0^2}$$

This expression gives the maximum altitute α, given the initial velocity v_0.

We now solve v_0 in terms of α:

$$Rv_0^2 = 2g\alpha R - \alpha v_0^2$$

$$2g\alpha R = v_0^2(R+\alpha)$$

$$v_0^2 = \frac{2\alpha g R}{R+\alpha}$$

$$v_0 = \sqrt{\frac{2\alpha g R}{R+\alpha}} = \sqrt{\frac{2gR}{R/\alpha + 1}}$$

$$\boxed{v_0(\alpha) = \sqrt{\frac{2gR}{1+R/\alpha}}}$$ This solves (b)

The escape velocity v_e is found by letting $\alpha \to \infty$, $\lim_{\alpha\to\infty} v_0(\alpha) = \sqrt{2gR}$. We plug g and the radius R of the earth to obtain $v_e \cong 11.1$ Km/sec