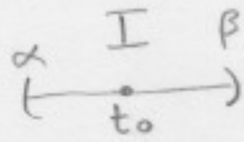
# Section 2.4
## Differences between linear and nonlinear differential equations.

<u>Theorem</u> Existence and uniqueness theorem for first-order linear equations:
If the functions $p$ and $g$ are continuous on an open interval $I: \alpha < t < \beta$ containing the point $t = t_0$ then there exists a unique function $y = \phi(t)$ that satisfies the differential equation:

$$y' + p(t)\,y = g(t) \qquad (**)$$

for each $t$ in $I$, and that also satisfies the initial condition




$$y(t_0) = y_0, \qquad (*)$$

where $y_0$ is an arbitrary prescribed initial value.

Sketch of proof: The proof of this theorem is partially contained in the derivation made in Section 2.1 (integrating factors). The derivation shows that if (*) has a solution, then it has to be of the form:

$$\mu(t)\,y = \int \mu(t)\,g(t)\,dt + C$$

$$\mu(t) = e^{\int p(t)\,dt}$$

Indeed, if $y(t)$, $y(t_0) = y_0$ is a solution of $(*)$ then:

$$y' + py = g$$

$$\Rightarrow \mu y' + \mu p y = \mu g$$

$$\Rightarrow \mu y' + \mu' y = \mu g \; ; \quad \text{if } \mu' = \mu p, \text{ or } \mu(t) = e^{\int_{t_0}^{t} p(s)\,ds}$$

$$\Rightarrow \frac{d}{dt}(\mu y) = \mu g$$

$$\Rightarrow \int_{t_0}^{t} \frac{d}{ds}\left(\mu(s) y(s)\right) = \int_{t_0}^{t} \mu(s) g(s)\, ds$$

Fundamental theorem of calculus yields:

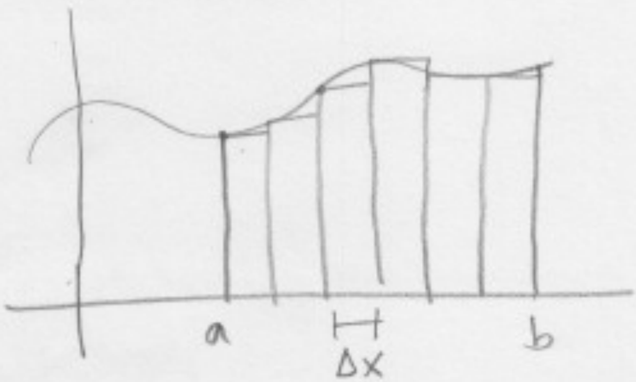
$$\mu(t) y(t) - \mu(t_0) y(t_0) = \int_{t_0}^{t} \mu g\, ds$$

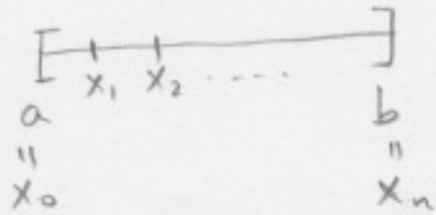
$$\Rightarrow \mu(t)\, y(t) = \int_{t_0}^{t} \mu g\, ds + y(t_0); \quad \text{since } \mu(t_0) = 1$$

$$(**) \quad y(t) = \frac{1}{\mu(t)} \int_{t_0}^{t} \mu g\, ds + \frac{y_0}{\mu(t)}, \quad \mu(t) = e^{\int_{t_0}^{t} p(s)\,ds}$$

Conversely, if $y(t)$ is of the form $(**)$, by taking derivatives we go back to the equation, showing that $y(t)$ is a solution to the equation.

<u>Remark</u>: The integrals $\int p\,dt$ and $\int g\mu\,dt$ exist because $p$ and $g$ are assumed to be continuous, and it is proven in an analysis class that the Riemman integral $\int f(x)dx$ exists if $f$ is continuous, as the limit of Riemann sums;




$$\int f(x)dx = \lim_{\substack{\Delta x \to 0 \\ (n \to \infty)}} \sum_{i=0}^{n-1} f(x_i)\Delta x$$

limit exists if $f$ is continuous

<u>Theorem</u> Existence and uniqueness theorem for first-order nonlinear equations.

Let the functions $f$ and $\frac{\partial f}{\partial y}$ be continuous in some rectangle $\alpha < t < \beta$, $\gamma < y < \delta$ containing the point $(t_0, y_0)$. Then, in some interval $t_0 - h < t < t_0 + h$ contained in $\alpha < t < \beta$, there is a unique solution $y = \phi(t)$ of the initial value problem

$$y' = f(t, y) \qquad y(t_0) = y_0$$

Discussion of Proof: Since there is no general formula for the solution of arbitrary nonlinear first order initial value problems, this proof is difficult, and is beyond the scope of this course.

It turns out that the conditions of the theorem are sufficient but not necessary to guarantee existence of a solution, and continuity of $f$ ensures existence but not uniqueness of the solution.

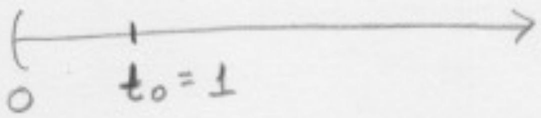
Ex: Verify the theorem of existence and uniqueness for the following first order linear equation:

$$ty' - 2y = 5t^2, \quad y(1) = 2$$

Solution:

$$y' - \frac{2}{t} = 5t$$

$p(t) = -\frac{2}{t}$ is continuous on $(0, \infty)$, which is $I$:




$g(t) = 5t$ is also continuous on $I$.

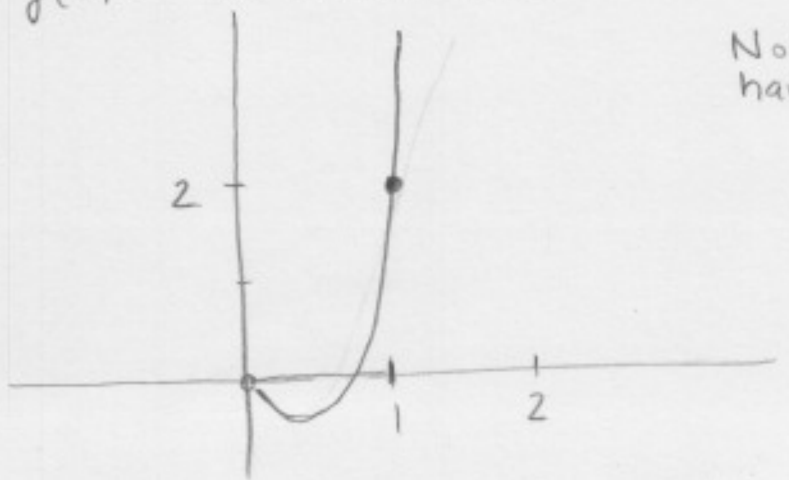
Hence the theorem applies and there exists a unique solution to the IVP. We can compute it:

$$\mu(t) = e^{\int -\frac{2}{t} dt} = e^{-2\ln|t|} = e^{\ln|t|^{-2}} = t^{-2} = \frac{1}{t^2}$$

$$\int \frac{d}{dt}\left(\frac{1}{t^2} y(t)\right) = \int \frac{5}{t}$$

$$\frac{1}{t^2} y(t) = 5\ln|t| + c, \quad y(1) = 2 \Rightarrow C = 2$$

$$\Rightarrow y(t) = 5t^2 \ln|t| + 2t^2.$$




Note: If we have the IVP

$$\begin{cases} ty' - 2y = 5t^2 \\ y(-1) = 2 \end{cases}$$

the same arguments apply with $I = (-\infty, 0)$

Ex: Consider the first order non-linear IVP:

$$y' = y^{1/3}, \quad y(0)=0, \quad t \geq 0.$$

Here, $f(t,y) = y^{1/3}$ which is continuous everywhere, but

$$\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3} = \frac{1}{3y^{2/3}}$$

$\frac{\partial f}{\partial y}$ is not continuous at $y=0$, and hence the hypothesis of the theorem for non-linear equations are not satisfied. Indeed, we can not find a rectangle containing $(0,0)$ where both $f$ and $\frac{\partial f}{\partial y}$ are continuous.

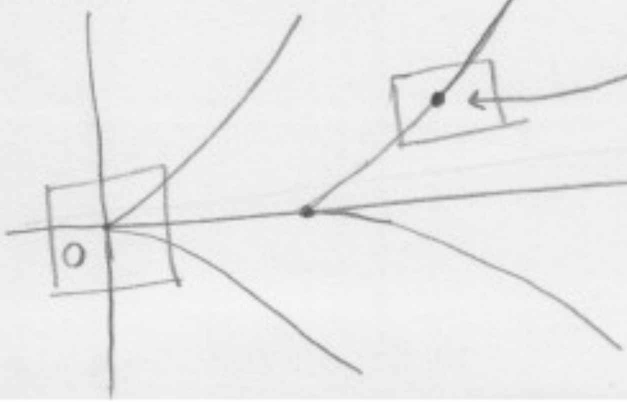
Solutions exist but are not unique. Separating variables we get:

$$\int \frac{dy}{y^{1/3}} = \int dt \quad \Rightarrow \quad \frac{y^{2/3}}{2/3} = t + c$$

$$\Rightarrow \quad y^{2/3} = \frac{2}{3}t + c, \quad y(0)=0 \Rightarrow c = 0$$

Hence: $y^2 = \left(\frac{2}{3}t\right)^3 \Rightarrow y = \pm\sqrt{\left(\frac{2}{3}t\right)^3}, \quad t \geq 0.$

Note: Two solutions starting at $(0,0)$




If the initial condition is not on t-axis, then the theorem guarantees existence and uniqueness.

Ex: Consider the first order non-linear IVP:

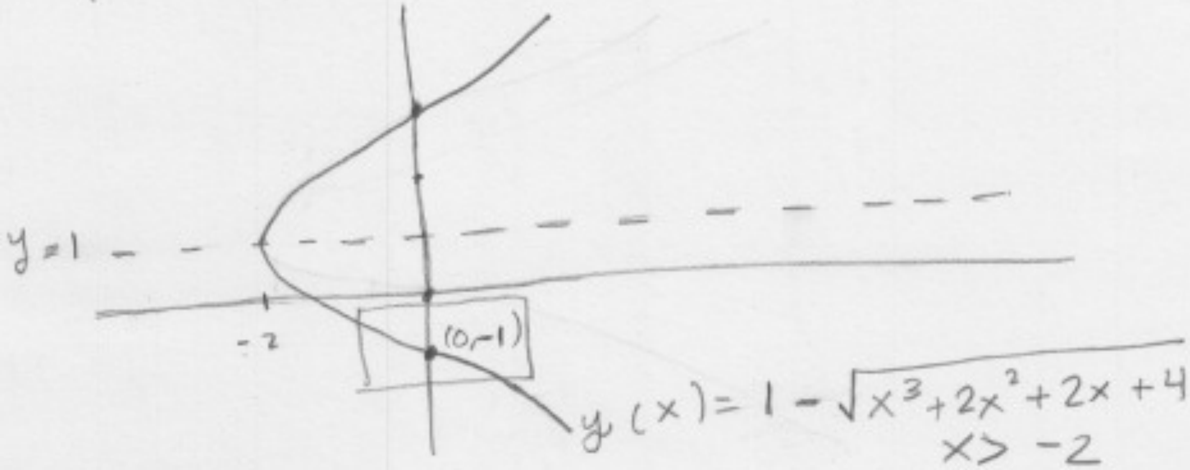
$$\frac{dy}{dx} = \frac{3x^2+4x+2}{2(y-1)} \qquad y(0) = -1$$

The functions:

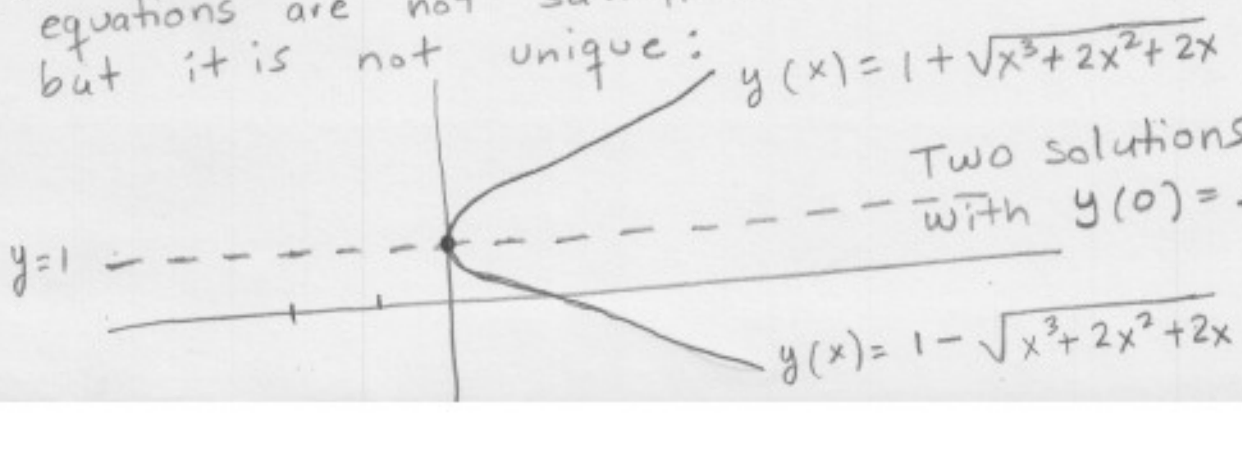
$$f(x,y) = \frac{3x^2+4x+2}{2(y-1)} \qquad \frac{\partial f}{\partial y}(x,y) = -\frac{3x^2+4x+2}{2(y-1)^2}$$

are continuous except on the line $y=1$.

Thus, we can draw an open rectangle about $(0,-1)$ on which $f$ and $\frac{\partial f}{\partial y}$ are continuous, as long as it doesn't cover $y=1$. Hence, the theorem applies and we have that existence of a unique solution with $y(0)=-1$:




If we change the initial condition to $y(0)=1$, then the hypothesis of the theorem of existence for non-linear equations are not satisfied. The solution exist but it is not unique:



Ex: Consider the nonlinear IVP:

$$y' = y^2, \quad y(0) = 1.$$

We have:

$$f(t,y) = y^2 \qquad \frac{\partial f}{\partial y} = 2y$$

Both functions are continuous everywhere. Clearly, there exist a rectangle around $(0,1)$ where both functions are continuous. Hence, the theorem of existence a uniqueness for non-linear first order equations yields the existence of a unique solution to $y' = y^2$ with $y(0) = 1$, which is defined at least in some interval $t_0 - h < t_0 < t_0 + h$.
The theorem does not give the size of $h$, only the existence of such $h$.
In this example we can actually solve the equation and find the domain of the solution.

$$y^{-2} dy = dt \Rightarrow -y^{-1} = t + c \Rightarrow y = \frac{-1}{t+c}$$

$$y(0) = 1 \Rightarrow y(t) = \frac{1}{1-t}, \text{ defined on } (-\infty, 1).$$