Section 2.5

Autonomous equations

An ODE is autonomous if it is of the form:

$$y'(t) = f(y(t)),$$

that is, f is a function of y only (not t).

Ex: $y' = y^2 + t^2$, $f(t,y) = y^2 + t^2$ is not autonomous

$y' = y^2$, $f(y) = y^2$ is autonomous.

Ex: Consider the logistic equation:

$$y'(t) = r\left(1 - \frac{y}{K}\right)y, \quad r, K > 0$$
$$= r\left(y - \frac{y^2}{K}\right)$$

Here, $f(y) = r\left(1 - \frac{y}{K}\right)y$.

The goal of this section is to sketch the solutions of an autonomous equation without solving the equation or using the computer. We will only use the equation to obtain information on the sign of $y'(t)$ and $y''(t)$. We will explain the method with the logistic equation.

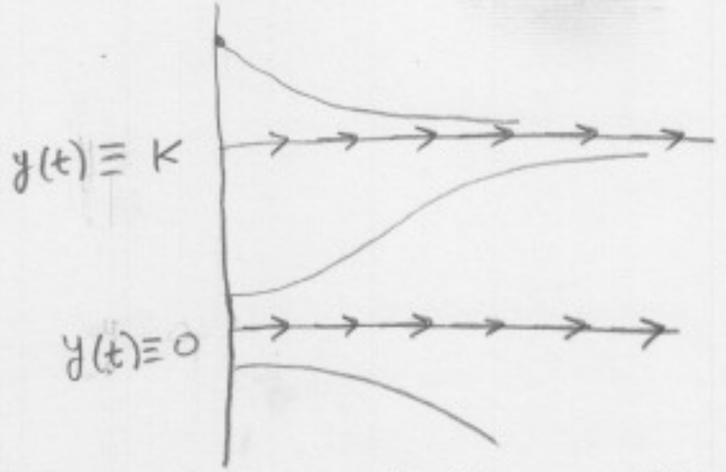
Step 1: The equilibrium solutions are the constant solutions of the equation:

$$f(y) = 0 \Rightarrow r\left(1 - \frac{y}{K}\right)y = 0$$

$y_1(t) \equiv 0$ and $y_2(t) \equiv K$ are the constant solutions.

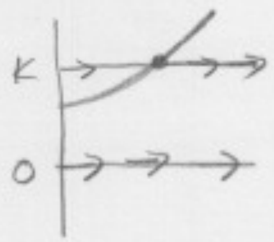
The roots of $f(y)=0$ are called critical points. We have 2 critical points:

$$y=0 \quad \text{and} \quad y=K.$$



Two constant solutions.

If $y(t)$ is a solution with initial condition $0<y(0)<K$, then $y(t)$ remains in the middle region between $y=0$ and $y=K$. This is because the existence and <u>uniqueness</u> theorem for non-linear ODE applies to $y'=r(1-\frac{y}{K})y$, and thus two solutions can not cross. If they could cross, we would not have uniqueness:



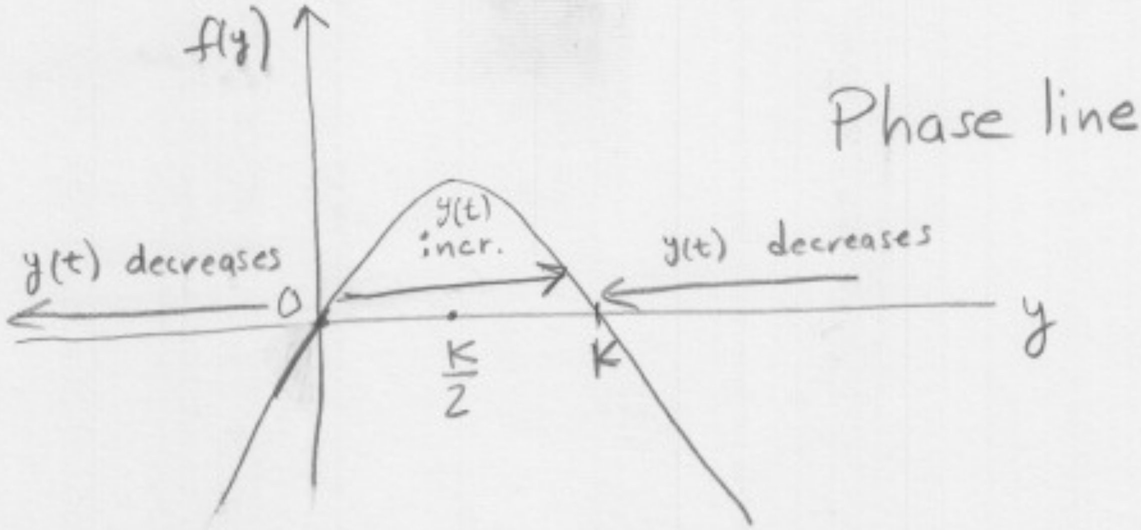
This can not happen.

The same is true if $y(0)<0$ or $y(0)>K$: the solution remains in region $y<0$ or $y>K$.

In order to be able to accurately draw the solutions $y(t)$ we need to know more about the shape of $y(t)$: where are they increasing or decreasing? what is their concavity?

<u>Step 2</u>: We graph y versus $f(y)$.

$$f(y) = r\left(y - \frac{y^2}{K}\right)$$



f has a maximum at $y = \frac{K}{2}$.

* If t is such that $0 < y(t) < K$, then, since $y'(t) = f(y(t))$, and f is positive for $0 < y < K$ we have that $y'(t) > 0$, that is $y(t)$ is increasing. We represent this reasoning in the above graph with the arrow $\longrightarrow$ going from 0 to K.

* If t is such that $y(t) < 0$ or $y(t) > K$, then, since $y'(t) = f(y(t))$ and f is negative for $y < 0$ or $y > K$ it follows that $y'(t) < 0$; that is $y(t)$ is decreasing.
To represent this is the graph, we draw arrows $\longleftarrow$.
This graph is called a phase line.

Step 3: Analysis of Concavities.

Since an homogeneous equation is of the form

$$y'(t) = f(y(t)),$$

we can differentiate on both sides of the equation to get:

$$y''(t) = \frac{d}{dt}\left(f(y(t)\right)$$
$$= f'(y(t)) \cdot y'(t) \quad ; \text{ using chain rule}$$
$$= f'(y(t)) f(y(t)) ; \quad \text{since } y' = f(y)$$

Hence:

$$\boxed{y''(t) = f'(y(t)) f(y(t))}$$

* Thus, the graph of $y(t)$ is concave up when f and f' have the same sign, which occurs when $0 < y(t) < \frac{K}{2}$ and $y(t) > K$

* The graph of $y(t)$ is concave down when f and f' have opposite signs, which occurs when $\frac{K}{2} < y(t) < K$ and $y(t) < 0$

We can make a diagram:

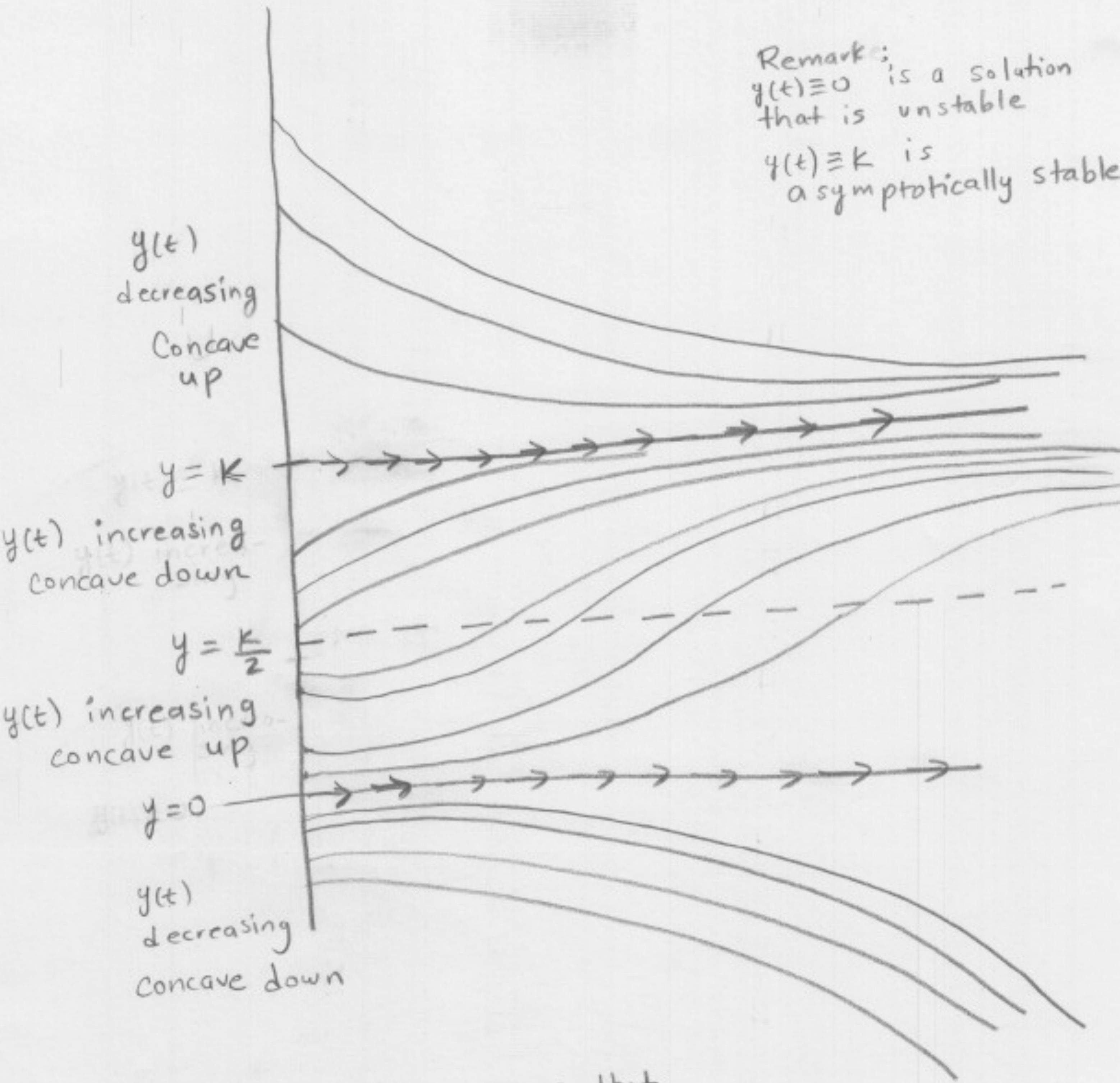
	$f'(y(t))$	$f(y(t)) = y'(t)$	$y''(t) = f'(y(t))f(y(t))$	
$y(t) > K$	$-$	$-$	$+$	$\cup$
$\frac{K}{2} < y(t) < K$	$-$	$+$	$-$	$\cap$
$0 < y(t) < \frac{K}{2}$	$+$	$+$	$+$	$\cup$
$y(t) < 0$	$+$	$-$	$-$	$\cap$

Step 4 :

We can finally sketch the solutions:

Remark: $y(t) \equiv 0$ is a solution that is unstable

$y(t) \equiv K$ is asymptotically stable



Note: Increasing means that the tangent lines to the curve have positive slope

Decreasing means that the tangent lines to the solution curve $y(t)$ have negative slope.

Ex: Consider the equation:

$$\frac{dy}{dt} = -y(1-y)\left(1-\frac{y}{2}\right)$$
$$= f(y)$$

$$f(y) = -y(1-y)\left(1-\frac{y}{2}\right)$$

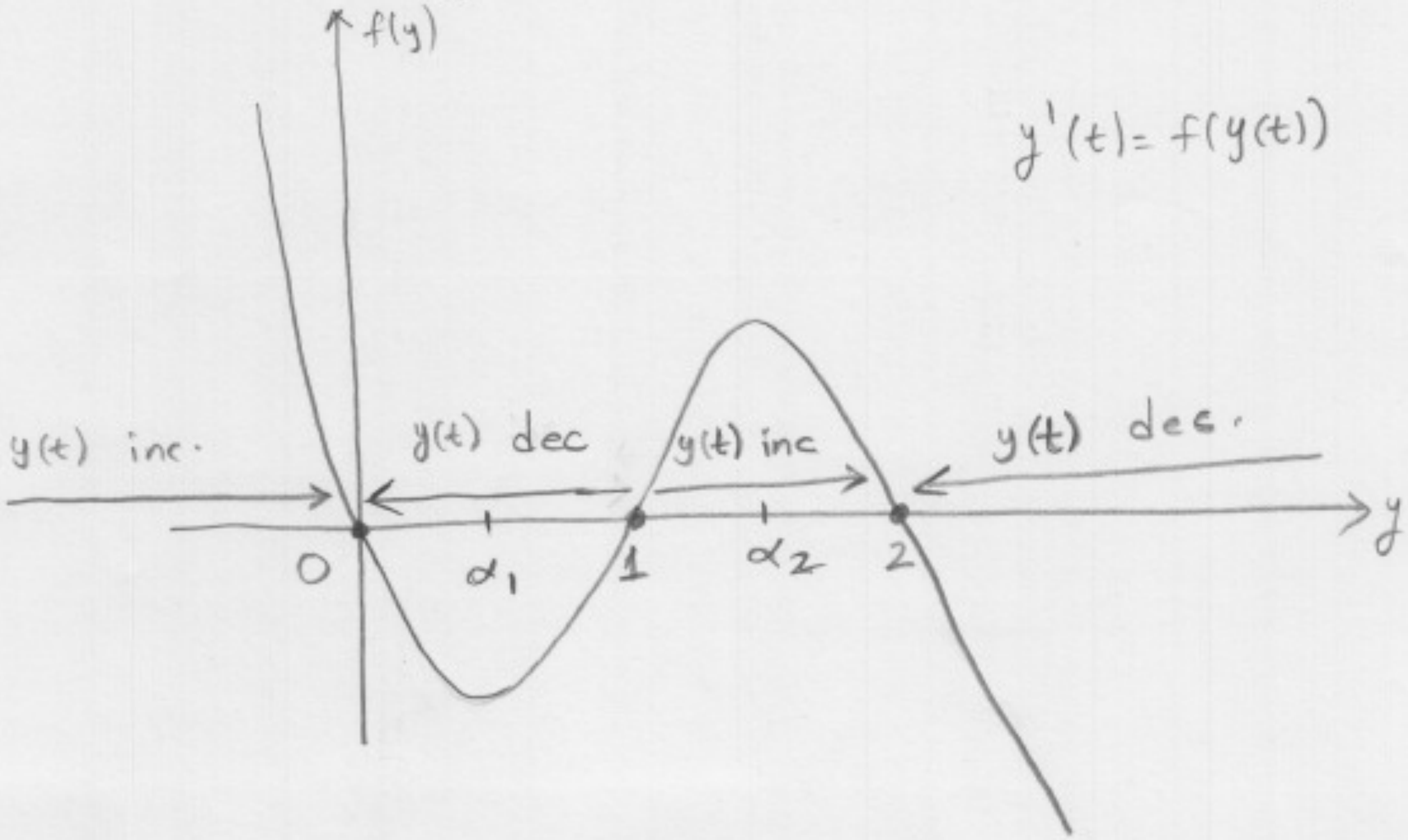
Step 1 : The critical points are:

$$y=0, \quad y=1, \quad y=2$$

Hence, the equilibrium solutions (or constant solutions) are:

$$y_1(t) \equiv 0, \quad y_2(t) \equiv 1, \quad y_3(t) \equiv 2.$$

Step 2 : We draw the Phase line; that is, we graph y versurs f(y) and find the ranges where the solutions y(t) are increasing or decreasing.

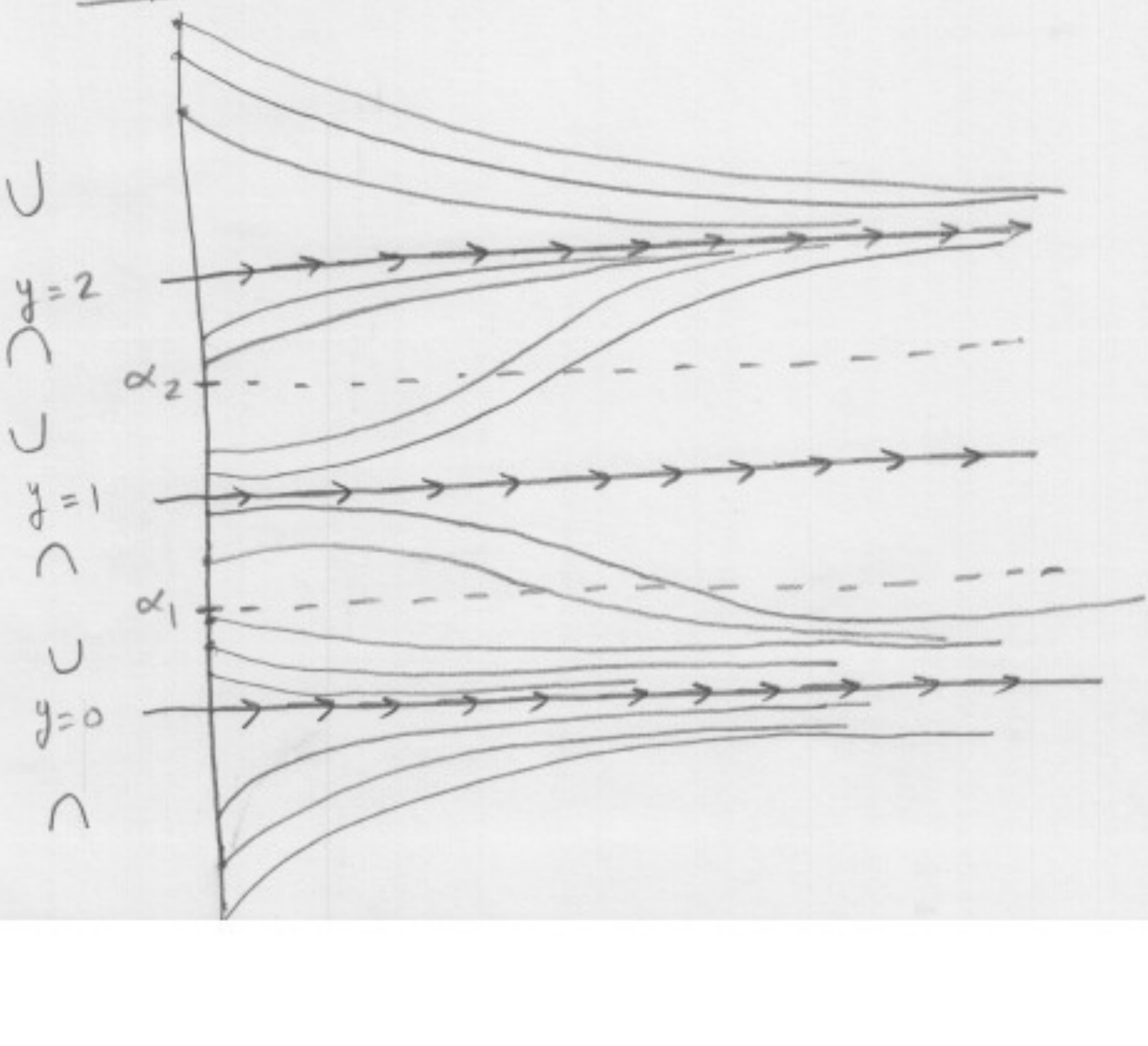


Step 3 : Analysis of concavities.

$$y''(t) = f'(y(t))f(y(t))$$

	$f'(y(t))$	$f(y(t)) = y'(t)$	$y''(t) = f'(y(t))f(y(t))$
$y(t) > 2$	$-$	$-$	$+$ $\cup$
$\alpha_2 < y(t) < 2$	$-$	$+$	$-$ $\cap$
$1 < y(t) < \alpha_2$	$+$	$+$	$+$ $\cup$
$\alpha_1 < y(t) < 1$	$+$	$-$	$-$ $\cap$
$0 < y(t) < \alpha_1$	$-$	$-$	$+$ $\cup$
$y(t) < 0$	$-$	$+$	$-$ $\cap$

Step 4 : sketch solutions.



Classification of Solutions.

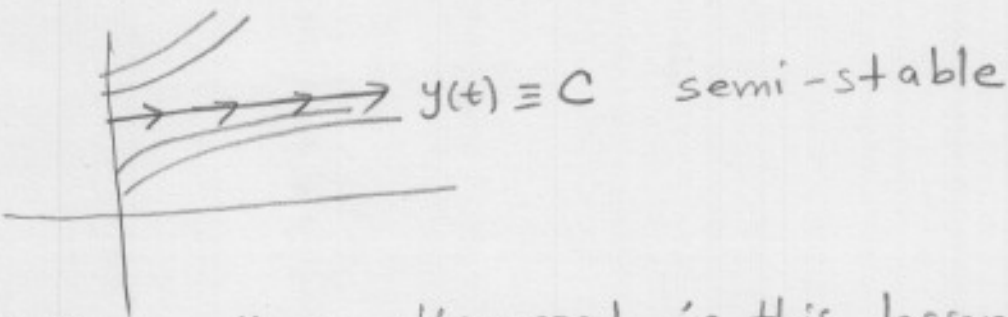
$y_1(t) \equiv 0$ and $y_3(t) \equiv 2$ are asymptotically stable

$y_2(t) \equiv 1$ is unstable.

y_1 (or y_3) is asymptotically stable because solutions that start with initial condition close to 0 (or 2) eventually, as $t \to \infty$, converge to y_1 (or y_3) <u>from both sides</u>.

$y_2(t)$ is unstable because solutions that start with initial condition close to 1 diverge, as $t \to \infty$, from the solution y_2, and this is true on both sides of the line.

An equilibrium solution $y(t) \equiv C$ is semi-stable if solutions starting with initial condition close to C converge to the solution (on one side of the line) and diverge from $y \equiv C$ on the other side.



<u>Note</u>: Both equations discussed in this lesson are separable. We can use partial fractions to perform the integration in y, and find formulas for solutions $y(t)$.