Section 2.6.

Consider a first order ODE of the form:

$$\boxed{M(x,y) + N(x,y)\,y' = 0 \qquad (*)}$$

Ex: Solve the equation:

$$\underbrace{(y\cos x + 2xe^{y})}_{M(x,y)} + \underbrace{(\sin x + x^2 e^{y} - 1)}_{N(x,y)}\,y' = 0$$

Suppose that there is a function $\psi(x,y)$ such that:

$$\frac{\partial \psi}{\partial x} = M(x,y), \quad \frac{\partial \psi}{\partial y}(x,y) = N(x,y)$$

In this case, we can re-write $(*)$ as:

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\cdot\frac{dy}{dx} = 0 \qquad (**)$$

Consider the composition $\psi(x, y(x))$.
We compute, using the chain rule:

$$\frac{d}{dx}\psi(x, y(x)) = \frac{\partial \psi}{\partial x}\cdot 1 + \frac{\partial \psi}{\partial y}\cdot\frac{dy}{dx}$$

Hence, the left part of $(**)$ becomes:

$$\frac{d}{dx}\left(\psi(x, y(x))\right) = 0$$

Integrating both sides with respect to x:

$$\int \frac{d}{dx}\psi(x, y(x))\,dx = \int 0\,dx$$

The fundamental theorem of calculus gives:

$$\boxed{\psi(x,y) = C}$$

Hence, the solution $y(x)$ is defined implicitly as:

$$\psi(x, y) = C.$$

Def: An ODE of the form (*) where there exists $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$ is called exact.

The following theorem gives the required condition for an ODE to be exact.

Theorem: There exists a function $\psi(x,y)$ such that $\frac{\partial \psi}{\partial x} = M$ and $\frac{\partial \psi}{\partial y} = N$ if and only if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Proof: the proof of this theorem is found in the ODE book. However, we can see this theorem as the 2-dimensional case of the following 3-dimensional theorem proven in Calculus III:

Thm: Let $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field. Then there exists a function $\psi(x,y,z)$ such that $F = \nabla \psi$ if and only if $\text{curl}\,\vec{F} = \vec{0}$.

Recall the definition of curl $\vec{F}$.

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{i}\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) - \vec{j}\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + \vec{k}\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

We apply this 3-d theorem to our ODE problem:

Let $\vec{F} = (M(x,y), N(x,y))$. Then there exits $\psi(x,y)$ such that $\vec{F} = \nabla\psi$ if and only if:

$$\text{Curl}\,\vec{F} = 0,$$

where

$$\text{Curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M(x,y) & N(x,y) & 0 \end{vmatrix} = \vec{i}(0) - \vec{j}(0) + \vec{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

Hence:

there exists $\psi(x,y)$ such that $(M,N) = \left(\frac{\partial \psi}{\partial x}, \frac{\partial \psi}{\partial y}\right)$ if and only if $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$; or $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$. ▫

We go back to our equation:

$$\underbrace{(y\cos x + 2xe^y)}_{M(x,y)} + \underbrace{(\sin x + x^2e^y - 1)}_{N(x,y)}\,y' = 0$$

$$\left.\begin{aligned} \frac{\partial M}{\partial y} &= \cos x + 2xe^y \\ \frac{\partial N}{\partial x} &= \cos x + 2xe^y \end{aligned}\right\} \frac{\partial M}{\partial y} = \frac{\partial M}{\partial x} \Rightarrow \text{ODE is exact.}$$

From theorem, $\exists \psi$ such that $\vec{F} = (M,N) = \nabla\psi$,

or:

$$\frac{\partial \psi}{\partial x} = y\cos x + 2xe^y = M$$

$$\frac{\partial \psi}{\partial y} = \sin x + x^2e^y - 1 = N$$

We find ψ by integrating

$$\Psi(x,y) = \int y\cos x + 2xe^{y}\,dx$$
$$= y\sin x + x^2e^y + g(y),$$

we add a function of y in order to have freedom and satisfy:

$$\frac{\partial\Psi}{\partial y} = \cancel{\sin x} + \cancel{x^2e^y} + g'(y) = N = \cancel{\sin x} + \cancel{x^2e^y} - 1$$

$$\Rightarrow \quad g'(y) = -1$$
$$\Rightarrow \quad g(y) = -y + C$$

We have:

$$\boxed{\Psi(x,y) = y\sin x + x^2e^y - y}$$

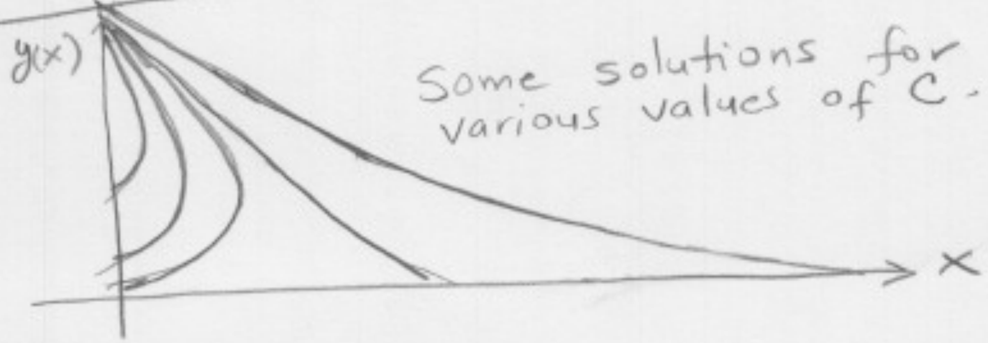
We double check:

$$\frac{\partial\Psi}{\partial x} = y\cos x + 2xe^y = M \checkmark$$
$$\frac{\partial\Psi}{\partial y} = \sin x + x^2e^y - 1 = N \checkmark$$

From our previous discussion we know that the solution $y(x)$ is given implicitly as:

$$\Psi(x,y) = C$$

or $\boxed{y\sin x + x^2e^y - y = C}$



Some solutions for various values of C.

Ex: Check our answer is correct:

We found:

$$y\sin x + x^2 e^y - y = c$$

to be the family of solutions of:

$$(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)\frac{dy}{dx} = 0$$

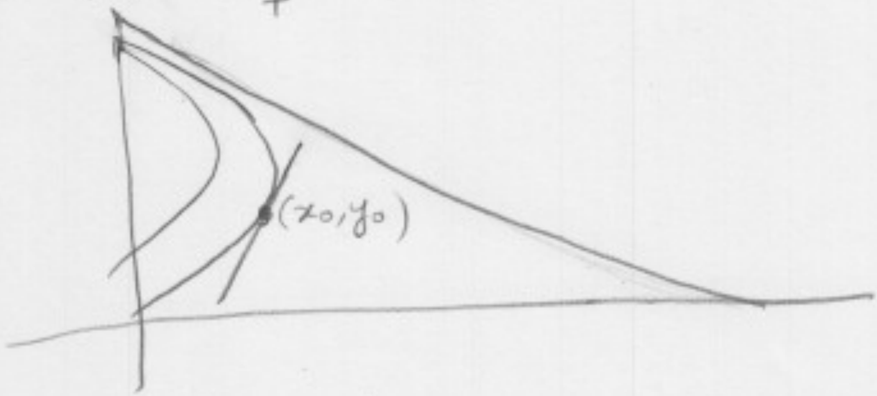
We differentiate implicitly with respect to x:

$$\frac{d}{dx}(y\sin x + x^2 e^y - y) = 0$$

$$y'\sin x + y\cos x + 2xe^y + x^2 e^y \cdot y' - y' = 0$$

$$\Rightarrow y'(\sin x + x^2 e^y - 1) + (y\cos x + 2xe^y) = 0,$$

which is the equation.



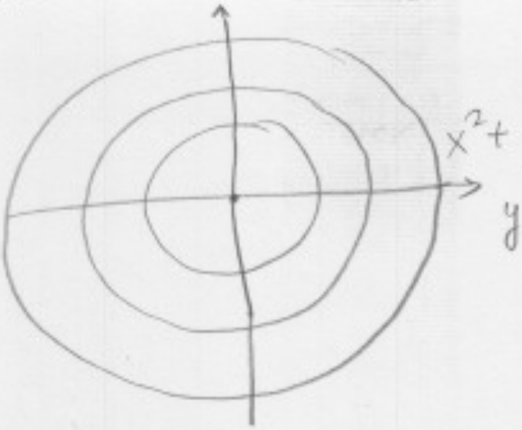
Our equation is:

$$y'(x) = -\frac{(y\cos x + 2xe^y)}{\sin x + x^2 e^y - 1}$$

The curve $y\sin x + x^2 e^y - y = C$ is a solution because, at (x_0, y_0), the slope of the tangent line to the curve is given by:

$$\frac{-y_0\cos x_0 + 2x_0 e^{y_0}}{\sin x_0 + x_0^2 e^{y_0} - 1},$$

Ex: Consider the family of circles $x^2+y^2=C$ (88)



$y(x)$ given implicitly

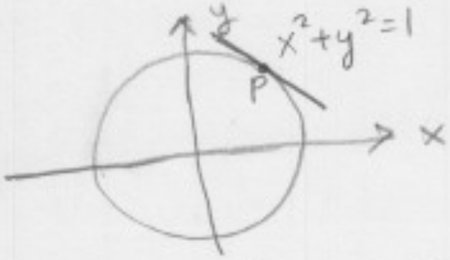
If we differentiate $x^2+y^2=C$ with respect to x we find the differential equation that these curves satisfy:

$$\frac{d}{dx}(x^2+y^2)=\frac{d}{dx}(C)$$

$$2x+2yy'=0$$

$$\Rightarrow \boxed{y'=-\frac{x}{y} \quad \text{or} \quad x+yy'=0} \quad \text{ODE}$$

If $C=1$, we have the solution curve $x^2+y^2=1$



$P=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ belongs to curve

we verified that $x^2+y^2=1$ is a solution curve to $x+y\frac{dy}{dx}=0$. Note that $P=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$, and at this point the slope of the tangent line is -1, which agrees with $y'\left(\frac{1}{\sqrt{2}}\right)=-\frac{x}{y}=\frac{-1/\sqrt{2}}{1/\sqrt{2}}=-1$.

Obviously, for this exact ODE, $\psi(x,y)=\frac{x^2}{2}+\frac{y^2}{2}$, which gives $\frac{x^2}{2}+\frac{y^2}{2}=C$, or $x^2+y^2=C$, the family of solutions.

Ex: Consider the following differential equation:

$$(3xy+y^2)+(x^2+xy)y'=0$$

In this case:

$$M(x,y)=3xy+y^2 \qquad N(x,y)=x^2+xy$$

$$\frac{\partial M}{\partial y}=3x+2y \qquad \frac{\partial N}{\partial x}=2x+y$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, hence it is not exact.

Question: Can we find a function $\mu(x)$ such that the equation

$$\mu(x)M(x,y)+\mu(x)N(x,y)y'=0$$

is exact? For this new equation to be exact, $\mu(x)$ must satisfy:

$$\frac{\partial}{\partial y}(\mu(x)M(x,y))=\frac{\partial}{\partial x}(\mu(x)N(x,y))$$

$$\mu(x)\frac{\partial M}{\partial y}=\mu(x)\frac{\partial N}{\partial x}+N\mu'(x)$$

$$N\mu'(x)=\mu(x)\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$$

$$\mu'(x)=\mu(x)\frac{(\partial M/\partial y-\partial N/\partial x)}{N}$$

In this problem:

$$\partial M/\partial y-\frac{\partial N}{\partial x}=3x+2y-2x-y=x+y$$

$$\Rightarrow \mu'(x) = \mu(x)\,\frac{\cancel{(x+y)}}{x\cancel{(x+y)}}$$

$$\Rightarrow \mu'(x) = \frac{\mu(x)}{x}$$

We can solve for $\mu(x)$ since equation is separable. Clearly, $\mu(x)=x$ is a solution, since:

$$\mu'(x) = 1 = \frac{\mu(x)}{x} = \frac{x}{x} = 1$$

Hence, with $\mu(x)=x$, the equation

$$(3xy+y^2) + (x^2+xy)y' = 0 \qquad (*)$$

can be made exact by multiplying by $\mu(x)=x$:

$$\underbrace{(3x^2y + xy^2)} + \underbrace{(x^3+x^2y)}\,y' = 0$$

$$\psi(x,y) = x^3y + \frac{x^2}{2}y^2 + g(y)$$

$$\frac{\partial\psi}{\partial y} = \cancel{x^3} + \cancel{x^2y} + g'(y) = \cancel{x^3} + \cancel{x^2y}$$

$$\Rightarrow g'(y) = 0 \quad \Rightarrow g(y) = C$$

Hence, solution $y(x)$ of $(*)$ is given implicitly by:

$$\psi(x,y) = c$$

$$\boxed{x^3y + \frac{x^2y^2}{2} = C}$$