Section 2.7

Numerical Approximations: Euler's method.

For our first order initial value problem:

$$(*)\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$$

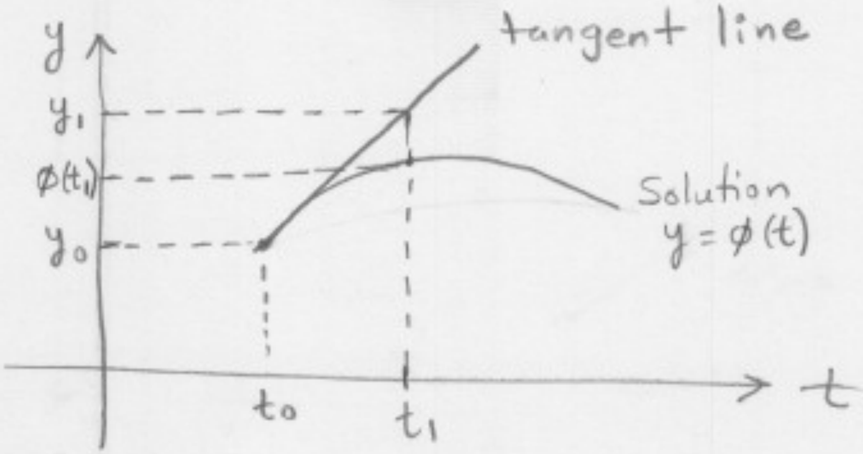
an alternative is to compute approximate values of the solution $y = \phi(t)$ at a selected set of t-values. Ideally, the approximate solution values will be accompanied by error bounds that ensure the level of accuracy.

There are many numerical methods that produce numerical approximations to solutions of differential equations, some of which are discussed in Chapter 8.

In this section, we examine the tangente line method, which is also called Euler's method.

For the IVP $(*)$ recall from that existence and uniqueness theorem that if f and $\partial f/\partial y$ are continuous in a rectangle containing (t_0, y_0), then there exists a unique solution $y = \phi(t)$ in some interval about t_0. When the differential equation is linear, separable or exact, we can find the solution by symbolic manipulations. However, the solutions for most differential equations of this form can not be found by analytical means.

Therefore, it is important to be able to approach the problem in other ways. In the Euler's method we begin as follows:



We have $\phi(t_0)=y_0$. We compute the equation of the tangent line to $\phi(t)$ at (t_0,y_0), which is:

$$y - y_0 = \text{slope } (t-t_0)$$

How do we compute the slope? The slope is $\phi'(t_0)$; notice that, since $\phi(t)$ solves the equation:

$$\phi'(t) = f(t,\phi(t)),$$

for every t in some interval $t \in (t_0-h, t_0+h)$, where h is given by the existence theorem. In particular:

$$\phi'(t_0) = f(t_0,\phi(t_0)) = f(t_0,y_0).$$

Hence, the slope $\phi'(t_0)$ is $f(t_0,y_0)$, and the tangent line is:

$$\boxed{y = y_0 + f(t_0,y_0)\,(t-t_0)}$$

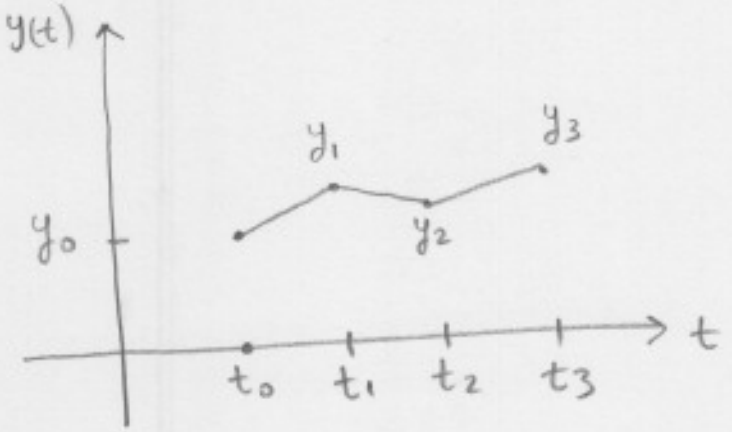
For t_1 close enough to t_0, we approximate $\phi(t_1)$ with the tangent line:

$$\boxed{\phi(t_1) \cong y(t_1) = y_0 + f(t_0, y_0)(t_1 - t_0)}$$

We denote $y(t_1)$ as y_1. Hence:

$$\boxed{\phi(t_1) \cong y_1}$$

We can move forward in time and approximate ϕ at t_2, t_3, ...



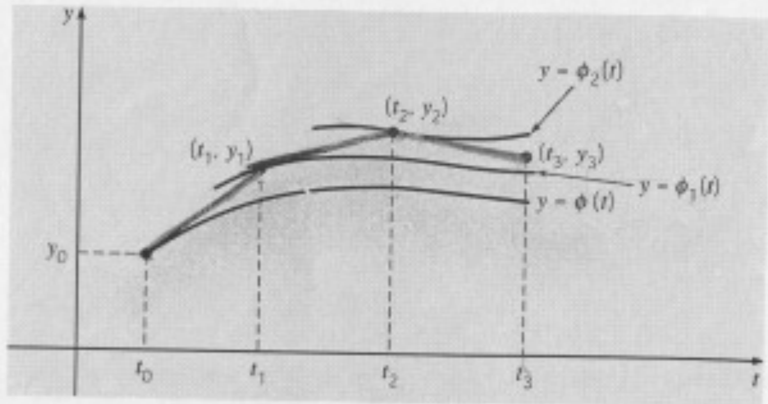
How do we approximate $\phi(t_2)$, $\phi(t_3)$ an so on?

The point (t_1, y_1) is typically not on the graph of ϕ, because y_1 is an approximation of $\phi(t_1)$. Hence, the next iteration of Euler's method does not use a tangent line approximation of ϕ, but rather to a nearby solution ϕ_1 that passes through the point (t_1, y_1). See picture in next page.

We use again the existence theorem to obtain that there exists a unique solution $y = \phi_1(t)$ that passes through the point (t_1, y_1); i.e; $\phi_1(t_1) = y_1$

General Error Analysis Discussion (2 of 4)

- The first step of Euler's method uses the tangent line to ϕ at the point (t_0, y_0) in order to estimate $\phi(t_1)$ with y_1.
- The point (t_1, y_1) is typically not on the graph of ϕ, because y_1 is an approximation of $\phi(t_1)$.
- Thus the next iteration of Euler's method does not use a tangent line approximation to ϕ, but rather to a nearby solution ϕ_1 that passes through the point (t_1, y_1).
- Thus Euler's method uses a succession of tangent lines to a sequence of different solutions $\phi, \phi_1, \phi_2, \ldots$ of the differential equation.



The equation of the tangent line to $\phi_1(t)$ at (t_1, y_1) is given by:

$$y - y_1 = \text{slope } (t - t_1), \quad \text{slope} = \phi_1'(t_1)$$

In order to compute the slope we note:

$\phi_1(t)$ is a solution to the equation, therefore:

$\phi_1'(t) = f(t, \phi_1(t))$ holds for every t in a neighborhood of t_1.

In particular it holds at t_1:

$$\phi_1'(t_1) = f(t_1, \phi_1(t_1)) = f(t_1, y_1).$$

The eq. of tangent line is:

$$\boxed{y = y_1 + f(t_1, y_1)(t - t_1)}$$

For t_2 close to t_1, we approximate $\phi(t_2)$ as:

$$\boxed{\phi(t_2) \cong y(t_2) = y_1 + f(t_1, y_1)(t_2 - t_1),}$$

and we denote this value as y_2. Hence

$$\boxed{\phi(t_2) \cong y_2}$$

We proceed in this way:

Existence theorem $\Rightarrow$ there exists a unique solution $y = \phi_2(t)$ that passes trough (t_2, y_2); i.e, $\phi_2(t_2) = y_2$. The equation of the tangent line to $\phi_2(t)$ at (t_2, y_2) is given by:

$$y - y_2 = \text{slope}\,(t - t_2), \quad \text{slope} = \phi_2'(t_2)$$

Since $\phi_2(t)$ is a solution $\Rightarrow$

$$\phi_2'(t) = f(t, \phi_2(t))$$

$$\Rightarrow \phi_2'(t_2) = f(t_2, \phi_2(t_2)) = f(t_2, y_2)$$

$\Rightarrow$ Tangent line is:

$$y = y_2 + f(t_2, y_2)(t - t_2).$$

We approximate $\phi(t_3)$ as:

$$\boxed{\begin{array}{l} \phi(t_3) \cong y_3 := y(t_3) = y_2 + f(t_2, y_2)(t_3 - t_2). \\ \Rightarrow \quad \phi(t_3) \cong y_3 \end{array}}$$

We proceed in this way to create a sequence y_n of approximations to $\phi(t_n)$:

$$y_1 = y_0 + f(t_0, y_0)(t_1 - t_0)$$
$$y_2 = y_1 + f(t_1, y_1)(t_2 - t_1)$$
$$y_3 = y_2 + f(t_2, y_2)(t_3 - t_2)$$
$$\vdots$$
$$y_{n+1} = y_n + f(t_n, y_n)(t_{n+1} - t_n)$$

For a uniform step size $h = t_{n+1} - t_n$, Euler's formula becomes:

$$y_{n+1} = y_n + f(t_n, y_n)h, \quad n = 0, 1, 2, \ldots$$

Example 1: Euler's Method (1 of 3)

- For the initial value problem

$$y' = 9.8 - 0.2y, \quad y(0) = 0 \qquad f(t,y) = 9.8 - 0.2y$$

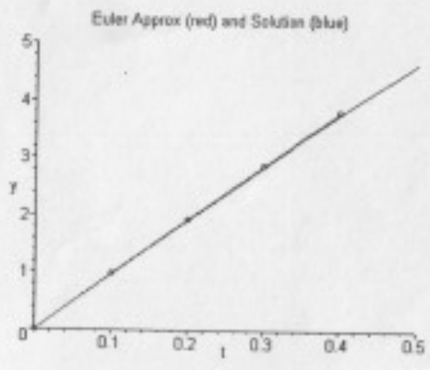
we can use Euler's method with $h = 0.1$ to approximate the solution at $t = 0.1, 0.2, 0.3, 0.4$, as shown below.

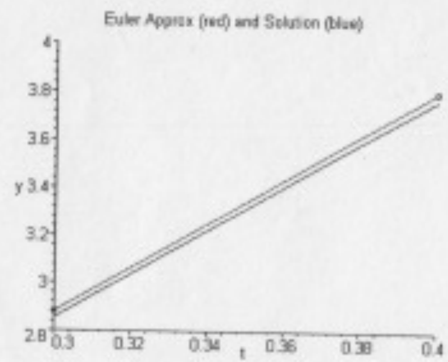
$$y_1 = y_0 + f_0 \cdot h = 0 + 9.8(0.1) = .98 \qquad f_n := f(t_n, y_n)$$

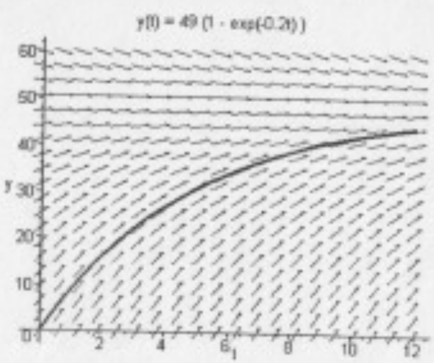
$$y_2 = y_1 + f_1 \cdot h = .98 + \underline{(9.8 - (0.2)(.98))}(0.1) \approx 1.94$$

$$y_3 = y_2 + f_2 \cdot h = 1.94 + \underline{(9.8 - (0.2)(1.94))}(0.1) \approx 2.88$$

$$y_4 = y_3 + f_3 \cdot h = 2.88 + \underline{(9.8 - (0.2)(2.88))}(0.1) \approx 3.80$$





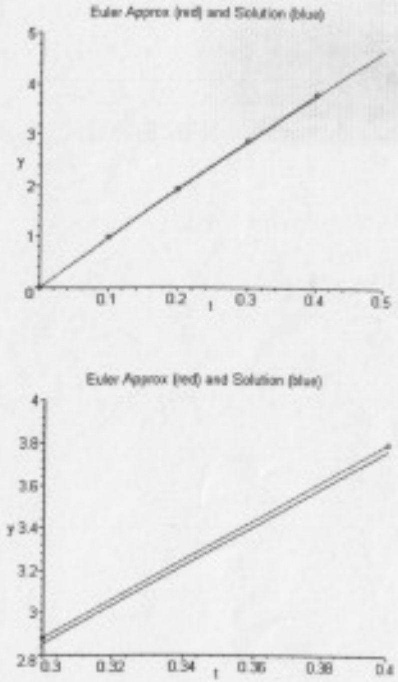


Example 1: Error Analysis (3 of 3)

* From table below, we see that the errors are small. This is most likely due to round-off error and the fact that the exact solution is approximately linear on [0, 0.4]. Note:

$$\text{Percent Relative Error} = \frac{y_{exact} - y_{approx}}{y_{exact}} \times 100$$

t	Exact y	Approx y	Error	% Rel Error
0.00	0	0.00	0.00	0.00
0.10	0.97	0.98	-0.01	-1.03
0.20	1.92	1.94	-0.02	-1.04
0.30	2.85	2.88	-0.03	-1.05
0.40	3.77	3.8	-0.03	-0.80



Example 2: Euler's Method (1 of 3)

※ For the initial value problem

$$y' = 4 - t + 2y, \quad y(0) = 1$$

$t_0 = 0$

$y_0 = 1$

we can use Euler's method with $h = 0.1$ to approximate the solution.

$$y_1 = y_0 + f_0 \cdot h = 1 + \left(4 - 0 + (2)(1)\right)(0.1) = 1.6$$

$$y_2 = y_1 + f_1 \cdot h = 1.6 + \left(4 - 0.1 + (2)(1.6)\right)(0.1) = 2.31$$

$$y_3 = y_2 + f_2 \cdot h = 2.31 + \left(4 - 0.2 + (2)(2.31)\right)(0.1) \approx 3.15$$

$$y_4 = y_3 + f_3 \cdot h = 3.15 + \left(4 - 0.3 + (2)(3.15)\right)(0.1) \approx 4.15$$

$$\vdots$$

※ Exact solution (see Chapter 2.1):

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

Example 2: Error Analysis (2 of 3)

- The first ten Euler approxs are given in table below on left. A table of approximations for $t = 0, 1, 2, 3$ is given on right. See text for numerical results with $h = 0.05, 0.025, 0.01$.
- The errors are small initially, but quickly reach an unacceptable level. This suggests a nonlinear solution.

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
0.10	1.66	1.60	0.06	3.55
0.20	2.45	2.31	0.14	5.81
0.30	3.41	3.15	0.26	7.59
0.40	4.57	4.15	0.42	9.14
0.50	5.98	5.34	0.63	10.58
0.60	7.68	6.76	0.92	11.96
0.70	9.75	8.45	1.30	13.31
0.80	12.27	10.47	1.80	14.64
0.90	15.34	12.89	2.45	15.96
1.00	19.07	15.78	3.29	17.27

t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Exact Solution:

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$

$y' = 4 - t + 2y \quad y(0) = 1.$

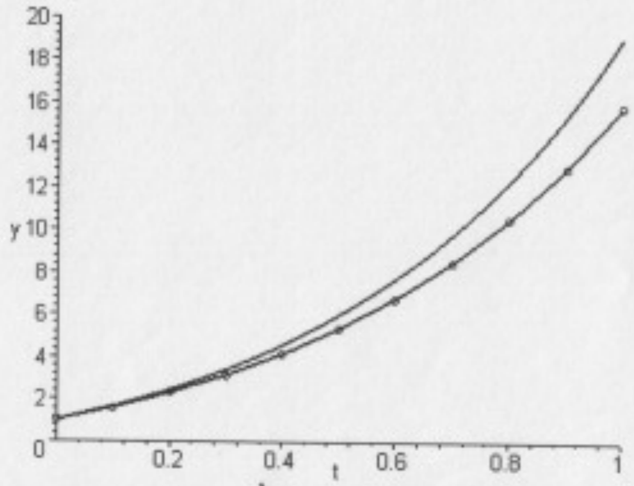
Example 2: Error Analysis & Graphs (3 of 3)

- Given below are graphs showing the exact solution (red) plotted together with the Euler approximation (blue).

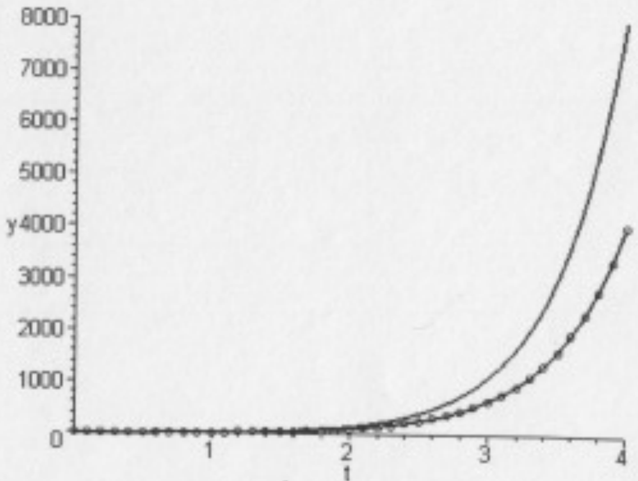
t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

Exact Solution :

$$y = -\frac{7}{4} + \frac{1}{2}t + \frac{11}{4}e^{2t}$$



h = 0.1



h = 0.1

Error Bounds and Numerical Methods

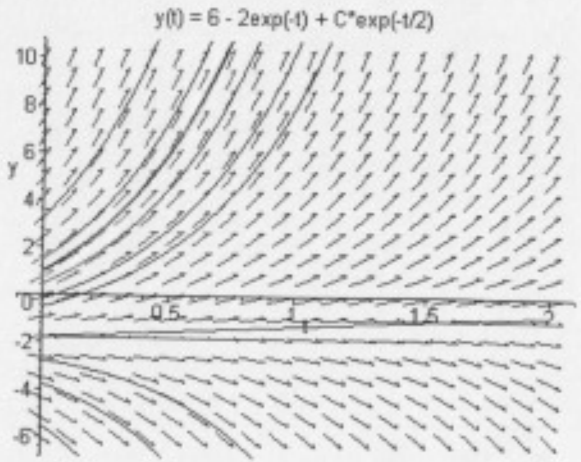
- In using a numerical procedure, keep in mind the question of whether the results are accurate enough to be useful.
- In our examples, we compared approximations with exact solutions. However, numerical procedures are usually used when an exact solution is not available. What is needed are bounds for (or estimates of) errors, which do not require knowledge of exact solution. More discussion on these issues and other numerical methods is given in Chapter 8.
- Since numerical approximations ideally reflect behavior of solution, a member of a diverging family of solutions is harder to approximate than a member of a converging family.
- Also, direction fields are often a relatively easy first step in understanding behavior of solutions.

Error Analysis Example: Divergent Family of Solutions (4 of 4)

- Now consider the initial value problem for Example 2:

$$y' = 4 - t + 2y, \;\; y(0) = 1 \;\Rightarrow\; y = -7/4 + t/2 + 11e^{2t}/4$$

- The direction field and graphs of solution curves are given below. Since the family of solutions is divergent, at each step of Euler's method we are following a different solution than the previous step, with each solution separating from the desired one more and more as t increases.

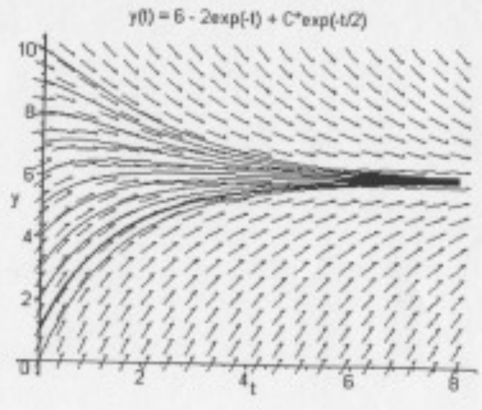


Error Analysis Example: Converging Family of Solutions (3 of 4)

- Since Euler's method uses tangent lines to a sequence of different solutions, the accuracy after many steps depends on behavior of solutions passing through (t_n, y_n), $n = 1, 2, 3, \ldots$
- For example, consider the following initial value problem:

$$y' = 3 + e^{-t} - y/2, \;\; y(0) = 1 \;\Rightarrow\; y = \phi(t) = 6 - 2e^{-t} - 3e^{-t/2}$$

- The direction field and graphs of a few solution curves are given below. Note that it doesn't matter which solutions we are approximating with tangent lines, as all solutions get closer to each other as t increases.
- Results of using Euler's method for this equation are given in text.



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