Section 3.1

Second order linear homogeneous equations with constant coefficients.

A second order differential equation has the general form:

$$y'' = f(t, y, y') \quad (*).$$

<u>Ex</u>: $y'' = t - yy'$ (non-linear)

The initial value problem requires two initial conditions:

$$\text{IVP} \begin{cases} y'' = f(t, y, y') \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$$

The equation $(*)$ is linear if and only if it is of the form:

$$P(t)\, y'' + Q(t)\, y' + R(t)\, y = G(t).$$

If $G(t) = 0$ for all t, then the equation is called homogeneous. Otherwise the equation is non-homogeneous.

We consider in this section the simplest case: when $G(t) \equiv 0$ and $P(t)=a$, $Q(t)=b$, and $R(t)=c$. The equation reduces to:

$$ay'' + by' + cy = 0 \rightarrow (**)$$

We look for solutions of the form:

$$y = e^{rt}$$

$$\Rightarrow y'(t) = re^{rt}, \quad y''(t) = r^2 e^{rt}$$

We plug in $(**)$ and look for r such that:

$$a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0$$

$$e^{rt}(ar^2 + br + c) = 0$$

Since e^{rt} is never 0, we need r such that:

$$ar^2 + br + c = 0$$

$$\Rightarrow r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \text{ two roots, } r_1, r_2$$

We have 3 cases:

Case 1: $r_1 \neq r_2$, real roots.

Case 2: $r_1 = r_2$ real number

Case 3: r_1, r_2 complex roots.

We consider in this section only
Case 1: $r_1 \neq r_2$.

In this case, we have two solutions:

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$$

We note that $c_1 y_1$ is also a solution, for any constant c_1, also $c_2 y_2$ is a solution for any constant c_2.
Moreover, any linear combination $c_1 y_1 + c_2 y_2$ is also a solution. We will see in section 3.2 that any solution to $ay'' + by' + cy = 0$ (case 1) is of the form $c_1 e^{r_1 t} + c_2 e^{r_2 t}$ and hence:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is the general solution to the equation.

<u>Ex</u>: Consider the IVP:

$$\begin{cases} y'' + 5y' + 6y = 0 \\ y(0) = 2, \quad y'(0) = 3. \end{cases}$$

$$r^2 + 5r + 6 = 0$$
$$(r+2)(r+3) = 0 \quad \Rightarrow \quad r_1 = -2, \; r_2 = -3$$

The general solution has the form:

$$y(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

We impose the initial conditions:

$y(0) = 2$ (i.e, $y(t)$ must pass through $(0,2)$)

$y'(0) = 3$ (i.e, the slope of the tangent line at $t=0$ must be 3).

$$y(0) = c_1 + c_2 = 2$$

$$y'(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}$$

$$y'(0) = -2c_1 - 3c_2 = 3.$$

We must solve:

$$\left.\begin{array}{l} c_1 + c_2 = 2 \\ -2c_1 - 3c_2 = 3 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 9 \\ c_2 = -7. \end{array}$$

Hence the particular solution that passes through $(0,2)$ and with slope 3 at $t=0$ is:

$$\boxed{y(t) = 9e^{-2t} - 7e^{-3t}}$$

Ex: Find the maximum value attained by the solution:

$$y(t) = 9e^{-2t} - 7e^{-3t}$$

$$y'(t) = -18e^{-2t} + 21e^{-3t} = 0$$

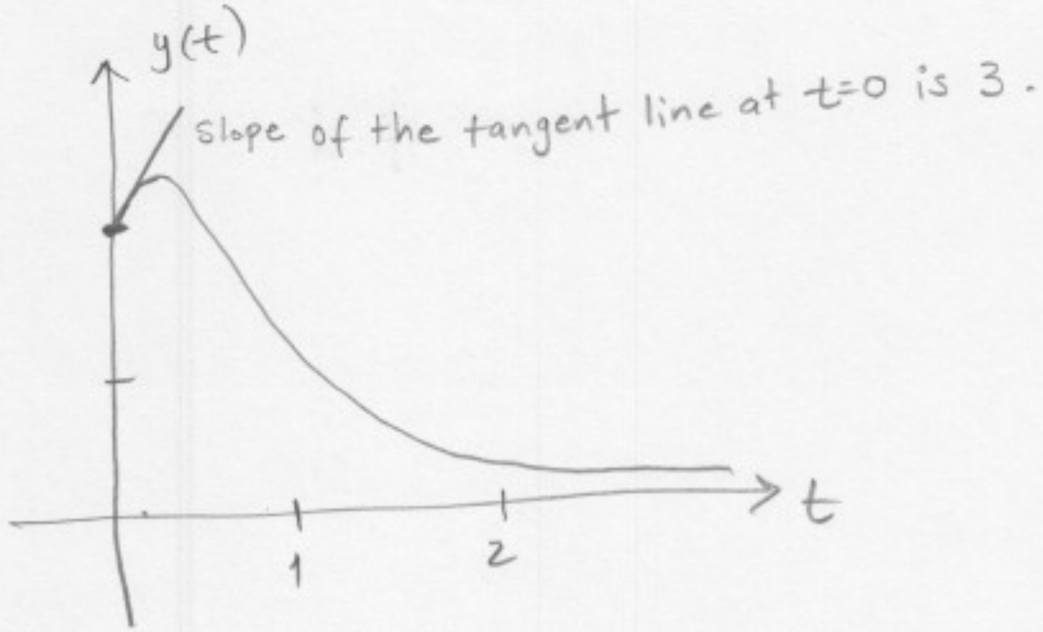
$$21e^{-3t} = 18e^{-2t}$$

$$7e^{-3t} = 6e^{-2t}$$

$$\frac{7}{6} = e^{t}$$

$$t = \ln(7/6) \cong 0.1542$$

$$y(0.1542) \cong 2.204$$



Note that $\lim_{t \to \infty} y(t) = 0$,