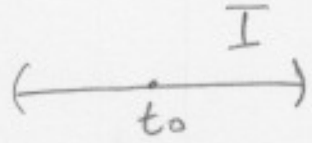
Section 3.2

Solutions of linear homogeneous second order equations: The Wronskian

<u>Existence theorem for linear second order equations</u>: Consider the initial value problem:

$$\begin{cases} y'' + p(t)y' + q(t)y = g(t) \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$$



where p, q and g are continuous on an open interval I that contains t_0. Then there exists a unique solution $y = \phi(t)$ on I.

Note: While this theorem says that a solution to the initial value problem above exists, it is often not possible to write down a useful expression for the solution. This is a major difference between first and second order linear equation. For first order linear:

$$\begin{cases} y' + p(t)y = g(t) \\ y(0) = y_0 \end{cases},$$

if $p(t)$ and $g(t)$ are continuous we can use the method of integrating factor to compute $y(t)$.

Note: Recall that, if $g(t)=0$, the equation is said to be homogeneous. If $g(t) \neq 0$, the equation is non-homogeneous.

Ex: Determine the longest interval on which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

$$(1+t)y'' - (\cos t)y' + 3y = 1, \quad y(0)=1, \quad y'(0)=0.$$

Solution: We first put the differential equation in to standard form:

$$y'' - \frac{\cos t}{1+t}y' + \frac{3}{1+t}y = \frac{1}{1+t}, \quad y(0)=1, \quad y'(0)=0$$

The longest interval containing the point $t=0$ on which the coefficient functions are continuous is $(-1, \infty)$. It follows from the existence theorem that the longest interval on which this initial value problem is certain to have a twice differentiable solution is also $(-1, \infty)$.

Principle of superposition

If y_1 and y_2 are two solutions of the differential equation:

$$y'' + p(t)y' + q(t)y = 0$$

then the linear combination $c_1y_1 + c_2y_2$ is also a solution for any values of the constants c_1 and c_2.

Indeed, since y_1 is a solution we have:

$$y_1'' + p(t)y_1' + q(t)y_1 = 0. \quad (1)$$

Since y_2 is also a solution we have:

$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \quad (2)$$

We plug $c_1y_1 + c_2y_2$ in the equation:

$$(c_1y_1 + c_2y_2)'' + p(t)(c_1y_1 + c_2y_2)' + q(t)(c_1y_1 + c_2y_2) =$$

$$c_1y_1'' + c_2y_2'' + p(t)(c_1y_1' + c_2y_2') + q(t)(c_1y_1 + c_2y_2) =$$

$$(c_1y_1'' + p(t)c_1y_1' + q(t)c_1y_1) + (c_2y_2'' + p(t)c_2y_2' + q\,c_2y_2) =$$

$$c_1(y_1'' + p(t)y_1' + q(t)y_1) + c_2(y_2'' + p(t)y_2' + q(t)y_2) =$$

$c_1(0) + c_2(0) = 0$; where we have used (1) and (2). Hence, $c_1y_1 + c_2y_2$ is a solution. ∎

Definition: If y_1 and y_2 are two solutions of:

$$y'' + p(t)y' + q(t)y = 0,$$

then the Wronskian of the solutions y_1 and y_2 at the point t_0 is defined as:

$$W(y_1, y_2)(t_0) = \det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix}$$

$$= \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

$$= y_1(t_0)\, y_2'(t_0) - y_1'(t_0)\, y_2(t_0)$$

Definiton: It is useful to define the differential operator $L(\phi)$ as follows:

$$L(\phi) = \phi'' + p\,\phi' + q\,\phi$$

Hence, if ϕ is a solution of $y'' + py' + qy = 0$, we have that:

$$\phi'' + p\phi' + q\phi = 0 \quad \text{or} \quad L(\phi) = 0.$$

We have seen that if we can find two solutions y_1 and y_2 of:

$$y'' + p(t)y' + q(t) = 0 , \quad (*)$$

then we actually have an infinite number of solutions. Indeed, the family of solutions $c_1 y_1 + c_2 y_2$ solves the equation. The question now is if this family includes ALL solutions to $(*)$. The answer is yes, if y_1 and y_2 satisfy a condition involving the Wronskian. Indeed, we have the following:

<u>Theorem**</u>: Suppose y_1 and y_2 are two solutions of the differential equation:

$$y'' + p(t)y' + q(t)y = 0 \quad (*),$$

where $p(t)$, $q(t)$ are continuous on the interval I. If there exists a point t_0 in I such that $W(y_1, y_2)(t_0) \neq 0$ then the family of solutions $c_1 y_1 + c_2 y_2$ with arbitrary coefficients c_1 and c_2 includes every solution.

<u>Proof</u>: Let ϕ be any solution of $(*)$. In order to prove the theorem we must determine whether ϕ is included in the linear

combination $c_1 y_1 + c_2 y_2$. That is, we must determine if we can find constansts α and β such that $\phi = \alpha y_1 + \beta y_2$.

The hypothesis is that we have t_0 in I with $W(y_1, y_2)(t_0) \neq 0$.

We define:

$$y_0 := \phi(t_0), \quad y_0' := \phi'(t_0)$$

We now consider the initial value problem:

$$(1) \begin{cases} y'' + p(t) y' + q(t) y = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_0' \end{cases}$$

Clearly, ϕ is a solution of IVP (1) since:

$$\phi'' + p(t)\phi' + q(t)\phi = 0, \quad \phi(t_0) = y_0, \quad \phi'(t_0) = y_0'$$

By the principle of superposition, $c_1 y_1 + c_2 y_2$ solves $(*)$ for any c_1, c_2. We now show that we can find α and β so that $\alpha y_1 + \beta y_2$ solves IVP (1). We impose the initial conditions:

$$\alpha y_1(t_0) + \beta y_2(t_0) = y_0$$
$$\alpha y_1'(t_0) + \beta y_2'(t_0) = y_0'$$

We solve for α and β:

$$\alpha y_1(t_0) y_1'(t_0) + \beta y_2(t_0) y_1'(t_0) = y_0 y_1'(t_0)$$
$$- \alpha y_1'(t_0) y_1(t_0) - \beta y_1(t_0) y_2'(t_0) = -y_0' y_1(t_0)$$
$$\beta \left[y_2(t_0) y_1'(t_0) - y_1(t_0) y_2'(t_0) \right] = y_0 y_1'(t_0) - y_0' y_1(t_0)$$

$$\Rightarrow \quad \beta = \frac{y_0\, y_1'(t_0) - y_0'\, y_1(t_0)}{y_2(t_0)\, y_1'(t_0) - y_1(t_0)\, y_2'(t_0)}$$

$$= \frac{y_0'\, y_1(t_0) - y_0\, y_1'(t_0)}{y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0)}$$

$$= \frac{y_0'\, y_1(t_0) - y_0\, y_1'(t_0)}{W(y_1, y_2)(t_0)}$$

$$y_1(t_0)\, \alpha = y_0 - \beta\, y_2(t_0)$$

$$= y_0 - y_2(t_0)\, \frac{y_0' y_1(t_0) - y_0 y_1'(t_0)}{y_1(t_0)\, y_2'(t_0) - y_2(t_0)\, y_1'(t_0)}$$

$$= \frac{y_0\, y_1(t_0) y_2'(t_0) \cancel{- y_0\, y_2(t_0) y_1'(t_0)} - y_2(t_0)\, y_0'\, y_1(t_0) \cancel{+ y_2(t_0) y_0\, y_1'(t_0)}}{W(y_1, y_2)(t_0)}$$

$$\Rightarrow \quad \alpha = \frac{y_0\, y_2'(t_0) - y_0'\, y_2(t_0)}{W(y_1, y_2)(t_0)}.$$

Since $W(y_1, y_2)(t_0) \neq 0$, we can solve and find a unique solution for α and β:

$$\alpha := \frac{y_0\, y_2'(t_0) - y_2(t_0) y_0'}{W(y_1, y_2)(t_0)}, \quad \beta := \frac{y_0'\, y_1(t_0) - y_0 y_1'(t_0)}{W(y_1, y_2)(t_0)}.$$

Therefore, the function $\alpha y_1 + \beta y_2$ is a solution to the initial value problem (1). Since ϕ is also a solution to the initial value problem (1) we conclude that both, ϕ and $\alpha y_1 + \beta y_2$, are solutions to the IVP (1).

The existence and uniqueness theorem in page 110 implies that:

$$\boxed{\phi = \alpha y_1 + \beta y_2},$$

since the solution to IVP (1) is <u>unique</u>.

We have shown that ϕ is contained in the family $c_1 y_1 + c_2 y_2$. ■

<u>Remark 1</u>: Recall that a linear system of n equations:

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n &= b_2 \\ &\vdots \\ a_{n1} x_1 + a_{n2} x_2 + \cdots + a_{nn} x_n &= b_n \end{aligned}$$

has a unique solution $\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ if and only if the matrix of coefficientes $A = (a_{ij})$ is invertible, i.e, $\det A \neq 0$. We can write the system as:

$$A\vec{x} = \vec{b}, \qquad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

If $\det A \neq 0$, A^{-1} exists. Hence:

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$
$$\Rightarrow \vec{x} = A^{-1}\vec{b},$$ hence $\vec{x}$ is unique.

In the previous theorem**, we can apply this remark to the system

$$\alpha y_1(t_0) + \beta y_2(t_0) = y_0$$
$$\alpha y_1'(t_0) + \beta y_2'(t_0) = y_0'$$

Then, we have a unique solution for α and β if and only if $\det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \neq 0$; i.e., $W(y_1, y_2)(t_0) \neq 0$.

<u>Remark 2</u>: If y_1 and y_2 satisfy Theorem**, then we say that $\{y_1, y_2\}$ form a fundamental set of solutions, and $y(t) = c_1 y_1 + c_2 y_2$ is <u>the general solution</u> of the differential equation $y'' + p(t)y' + q(t)y = 0$.

We have another theorem:

<u>Theorem</u> (Abel's theorem). If y_1 and y_2 are solutions of $y'' + p(t)y' + q(t)y = 0$ where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ has the formula:

$$W(y_1, y_2)(t) = c\, e^{-\int p(t)\,dt},$$

where c is a constant that depends on y_1 and y_2, but not on t.

Indeed, we check this theorem is true: since y_1 and y_2 are solutions we have:

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \quad (1)$$

$$y_2'' + p(t)y_2' + q(t)y_2 = 0 \quad (2)$$

We multiply (1) by $-y_2$, multiply (2) by y_1, and add the equations we obtain:

$$-y_2 y_1'' - p(t)y_2 y_1' - \cancel{q(t)y_2 y_1} = 0$$

$$y_1 y_2'' + p(t)y_1 y_2' + \cancel{q(t)y_2 y_1} = 0$$

$$(y_1 y_2'' - y_2 y_1'') + p(t)(y_1 y_2' - y_2 y_1') = 0 \quad (3)$$

We recall:

$$W(y_1, y_2)(t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2,$$ thus we have

that (setting $W(t) := W(y_1, y_2)(t)$):

$$W'(t) = y_1 y_2'' + \cancel{y_2' y_1'} - \cancel{y_1' y_2'} - y_1'' y_2$$

$$= y_1 y_2'' - y_1'' y_2$$

Hence, we write (3) as:

$$W'(t) + p(t) W(t) = 0. \qquad (4)$$

Note that (4) is a first order linear separable equation:

$$\frac{W'(t)}{W(t)} = -p(t)$$

$$\Rightarrow \ln |W(t)| = -\int p(t)\, dt$$

$$\Rightarrow W(t) = c\, e^{-\int p(t)\, dt} \quad \blacksquare$$

<u>Remark</u>: If $c = 0$, then $W(y_1, y_2)(t)$ is zero for all t in I. If $c \neq 0$, then $W(y_1, y_2)(t)$ is never zero in I.

Ex: We now go back to section 3.1, where we considered the equation:

$$ay'' + by' + cy = 0 \quad \text{Case 1: } r_1 \neq r_2 \text{ real roots of } ar^2 + br + c = 0$$

We found two solutions:

$$y_1(t) = e^{r_1 t}, \quad y_2(t) = e^{r_2 t}$$

We compute $W(y_1, y_2)(t)$:

$$W(y_1, y_2)(t) = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix}$$

$$= r_2 e^{(r_1+r_2)t} - r_1 e^{(r_1+r_2)t}$$

$$= e^{(r_1+r_2)t}(r_2 - r_1) \neq 0; \text{ because } r_1 \neq r_2$$

and the exponential is never 0.

Hence, Theorem ** implies that:

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is the general solution to the equation $ay'' + by' + cy = 0$ (case 1 when $r_1 \neq r_2$).

For this equation, the existence and uniqueness theorem applies to $I = (-\infty, \infty)$. Note that Wronskian is non-zero for all t in I.

Ex: Consider the following differential equation:

$$2t^2 y'' + 3t y' - y = 0, \quad t > 0$$

Show that the functions $y_1 = t^{1/2}$, $y_2 = t^{-1}$ form a fundamental set of solutions.

Solution: We first show that y_1, y_2 are indeed solutions to the equation:

$$y_1(t) = t^{1/2}, \quad y_1'(t) = \frac{1}{2} t^{-1/2} \quad y_1''(t) = -\frac{1}{4} t^{-3/2}$$

We plug in the equation:

$$2t^2 y_1'' + 3t y_1' - y_1 = 2t^2 \left(-\frac{1}{4} t^{-3/2}\right) + 3t \left(\frac{1}{2} t^{-1/2}\right) - t^{1/2}$$
$$= -\frac{1}{2} t^{1/2} + \frac{3}{2} t^{1/2} - t^{1/2} = 0,$$

hence y_1 is a solution

$$y_2(t) = t^{-1}, \quad y_2'(t) = -t^{-2}, \quad y_2''(t) = 2t^{-3}$$

We plug:

$$2t^2 y_2'' + 3t y_2' - y_2 = 2t^2 (2t^{-3}) + 3t(-t^{-2}) - t^{-1}$$
$$= 4t^{-1} - 3t^{-1} - t^{-1} = 0,$$

hence y_2 is a solution.

The canonical form of the equation is:

$$y'' + \underbrace{\frac{3}{2t}}_{p(t)} y' \underbrace{- \frac{1}{2t^2}}_{q(t)} y = 0$$

Since p, q are both continuous on $(0,\infty)$, then the initial value problem

$$\begin{cases} y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0 \\ y(t_0) = y_0, \quad y'(t_0) = y_0', \end{cases}$$

has a unique solution (t_0 belongs to $(0,\infty)$).

With $I = (0,\infty)$, we compute $W(y_1,y_2)(t)$, for t in I:

$$W(y_1,y_2)(t) = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2}$$

$$= -\frac{3}{2}t^{-3/2} = -\frac{3}{2\sqrt{t^3}}$$

$W(y_1,y_2)(t) \neq 0$ for t in $(0,\infty)$. Recall here Abel's theorem: either Wronskian is zero in whole I or is non-zero in the whole interval I.

We now apply Theorem ** to conclude that $\{t^{1/2}, t^{-1}\}$ is a fundamental set of solutions and that:

$$y(t) = c_1 t^{1/2} + c_2 t^{-1}$$

is the general solution to $2t^2y'' + 3ty' - y = 0$ with $I = (0,\infty)$. Any solution to this equation is contained in this family of solutions. ∎