Section 3.3

Complex roots of characteristic equation.

Recall our discussion of the equation:

$$ay'' + by' + cy = 0,$$

where a, b and c are constants

We look for solutions of the form $y(t) = e^{rt}$:

$$y'(t) = re^{rt}, \quad y''(t) = r^2 e^{rt}$$

We plug in equation:

$$a(r^2 e^{rt}) + b(re^{rt}) + ce^{rt} = 0$$

$$\Rightarrow \quad e^{rt}(ar^2 + br + c) = 0$$

$$\Rightarrow \quad ar^2 + br + c = 0$$

$$\Rightarrow \quad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We have 3 cases:

Case 1: $r_1 \neq r_2$ real solutions

Case 2: Complex roots, $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$

Case 3: $r_1 = r_2$ real solution

We have completed Case 1. The Wronskian Theorem discussed in Section 3.2 guarantees that $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is <u>the</u> general solution of $ay'' + by' + cy = 0$, case 1. That is, every solution is contained in this family.

In this section we discuss Case 2.
We have two complex solutions:

$$y_1(t) = e^{(\lambda + i\mu)t}, \quad y_2(t) = e^{(\lambda - i\mu)t}.$$

<u>Ex</u>: Consider the equation:

$$y'' + y' + y = 0$$

The characteristic equation is:

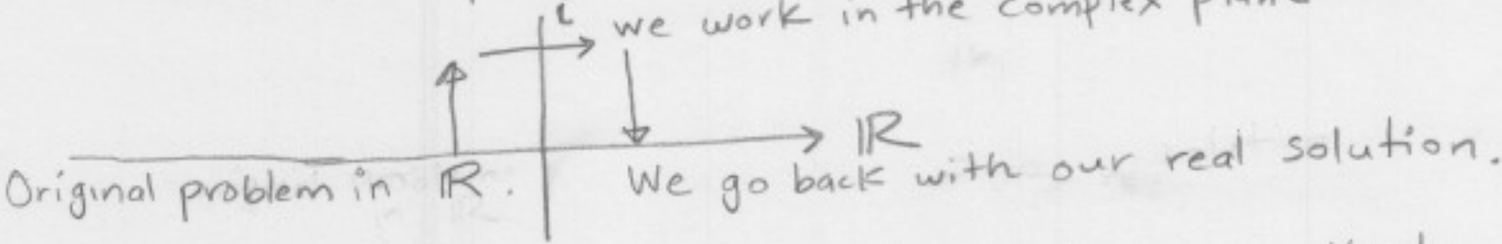
$$r^2 + r + 1 = 0$$

$$\Rightarrow r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$

We have two complex solutions:

$$y_1(t) = e^{\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)t}, \quad y_2(t) = e^{\left(-\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)t}$$

We recall that $i = \sqrt{-1}$, and: $i^2 = -1$, $i^3 = i^2 \cdot i = -i$, $i^4 = i^2 \cdot i^2 = 1$, a so on. We let $\sqrt{-1}$ to exist as an imaginary number and we follow the laws of algebra in the complex word. We expand the theory of calculus to the complex plane in more advanced classes.



We need to find two <u>real</u> solutions so that we can apply the Wronskian Theorem from previous section.

We double check that $y_1 = e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}$ solves $y''+y'+y=0$:

$$y_1'(t) = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}$$

$$y_1''(t) = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2 e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}$$

We plug:

$$y_1''+y_1'+y_1 = \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2 e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t} + \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t} + e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}$$

$$= e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}\left[\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^2 + \left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) + 1\right]$$

$$= e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}\left[\frac{1}{4} - \frac{2\sqrt{3}i}{4} + \frac{3}{4}i^2 - \frac{1}{2} + \frac{\sqrt{3}}{2}i + 1\right]$$

$$= e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}\left[\frac{1}{4} - \cancel{\frac{\sqrt{3}}{2}i} - \frac{3}{4} - \frac{1}{2} + \cancel{\frac{\sqrt{3}}{2}i} + 1\right]$$

$$= e^{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)t}\left[-\frac{1}{2} - \frac{1}{2} + 1\right] = 0.$$

In the same way we check that $y_2(t) = e^{\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)t}$ solves $y_2''+y_2'+y_2=0$.

Notice that $c_1 y_1$ is also a solution, where c_1 is any complex number:

$$(c_1 y_1)'' + (c_1 y_1)' + c_1 y_1 = c_1\left(y_1''+y_1'+y_1\right) = 0.$$

Also, y_1+y_2 is a solution since:

$$(y_1+y_2)'' + (y_1+y_2)' + (y_1+y_2) = (y_1''+y_1'+y_1) + (y_2''+y_2'+y_2) = 0+0=0$$

Indeed, any linear combination $c_1y_1+c_2y_2$ is a complex solution to $y''+y'+y=0$, where c_1 and c_2 are any complex numbers.

Recall the principle of <u>superposition</u>:
If y_1, y_2 are two solutions of $y''+py'+qy=0$, (second order, linear, homogeneous equation), then any linear combination $c_1y_1+c_2y_2$ is also a solution. This principle is also true for <u>complex</u> solutions, <u>and we need two real solutions</u>.

In complex analysis, the theory of calculus is extended: e^z, $\sin z$, $\cos z$ and other functions need to be defined, where z is a complex number.
It is proven in complex analysis that:

$$e^{i\theta} = \cos\theta + i\sin\theta, \text{ this is Euler's formula.}$$

Hence:

$$y_1(t) = e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}(\cos\mu t + i\sin\mu t)$$

$$y_2(t) = e^{(\lambda-i\mu)t} = e^{\lambda t} e^{-i\mu t} = e^{\lambda t}(\cos(-\mu t) + i\sin(-\mu t))$$

$$= e^{\lambda t}(\cos\mu t - i\sin\mu t)$$

In order to produce a real solution, we apply the principle of superposition. We add y_1+y_2:

$\tilde{y}_3 := y_1(t)+y_2(t) = 2e^{\lambda t}\cos\mu t$ is a solution

$y_3 := \frac{1}{2}\tilde{y}_3 = \frac{1}{2}(2e^{\lambda t}\cos\mu t) = e^{\lambda t}\cos\mu t$ is also a solution to $ay''+by'+cy=0$ Case 2.

We now substract y_1-y_2:
$\tilde{y}_4 := y_1-y_2 = 2i\, e^{\lambda t}\sin\mu t$ is also a solution.

$$y_4 := \frac{1}{2i}\tilde{y}_4 = \frac{1}{2i}\left(2i\, e^{\lambda t}\sin \mu t\right) = e^{\lambda t}\sin \mu t,$$

is also a solution to $ay''+by'+cy=0$, case 2.

We have found two <u>real</u> solutions:

$$\boxed{y_3(t) = e^{\lambda t}\cos \mu t, \quad y_4(t) = e^{\lambda t}\sin \mu t}$$

We compute the Wronskian of y_3 and y_4:

$$W(y_3,y_4)(t) = \begin{vmatrix} e^{\lambda t}\cos \mu t & e^{\lambda t}\sin \mu t \\ -\mu e^{\lambda t}\sin \mu t + \lambda e^{\lambda t}\cos \mu t & \mu e^{\lambda t}\cos \mu t + \lambda e^{\lambda t}\sin \mu t \end{vmatrix}$$

$$= e^{2\lambda t}\mu \cos^2 \mu t + \cancel{\lambda e^{2\lambda t}\sin \mu t \cos \mu t} + \mu e^{2\lambda t}\sin^2 \mu t - \cancel{\lambda e^{2\lambda t}\sin \mu t\cos \mu t}$$

$$= \mu e^{2\lambda t}\left(\sin^2 \mu t + \cos^2 \mu t\right)$$

$$= \mu e^{2\lambda t} \neq 0 \text{ for every } t, \text{ since } \mu \neq 0 \text{ (Case 2)}.$$

The Wronskian theorem implies that:

$$y(t) = c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t$$

is the general solution to $ay''+by'+cy=0$, Case 2.

<u>Note</u>: We needed two <u>real</u> solutions, we went to the complex word to solve the problem. We went back to $\mathbb{R}$ to apply our Wronskian theorem. We are done with this equation $ay''+by'+cy=0$, case 2.

Ex: Consider the equation:

$$y'' + 4y = 0$$

Find the general solution:

$$r^2 + 4 = 0$$
$$r^2 = -4 \Rightarrow r_1 = 2i, \; r_2 = -2i$$

$\Rightarrow$ $y(t) = C_1 \cos 2t + C_2 \sin 2t$ is the general solution, from previous discussion. We double check that $y_1(t) = \cos 2t$, $y_2(t) = \sin 2t$ are solutions:

$y_1(t) = \cos 2t$, $y_1'(t) = -2\sin 2t$, $y_1''(t) = -4\cos 2t$

$$y_1'' + 4y_1 = -4\cos 2t + 4\cos 2t = 0$$

$y_2(t) = \sin 2t$, $y_2'(t) = 2\cos 2t$, $y_2''(t) = -4\sin 2t$

$$y_2'' + 4y_2 = -4\sin 2t + 4\sin 2t = 0.$$

Ex: $y'' + y' + y = 0$

$$r^2 + r + 1 = 0$$

$$r = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i, \quad \lambda = -\frac{1}{2}, \quad \mu = \sqrt{3}/2$$

$$y_1(t) = e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2} t, \quad y_2(t) = e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t$$

The general solution to the equation is:

$$y(t) = c_1 e^{-\frac{t}{2}} \cos \frac{\sqrt{3}}{2} t + c_2 e^{-\frac{t}{2}} \sin \frac{\sqrt{3}}{2} t.$$

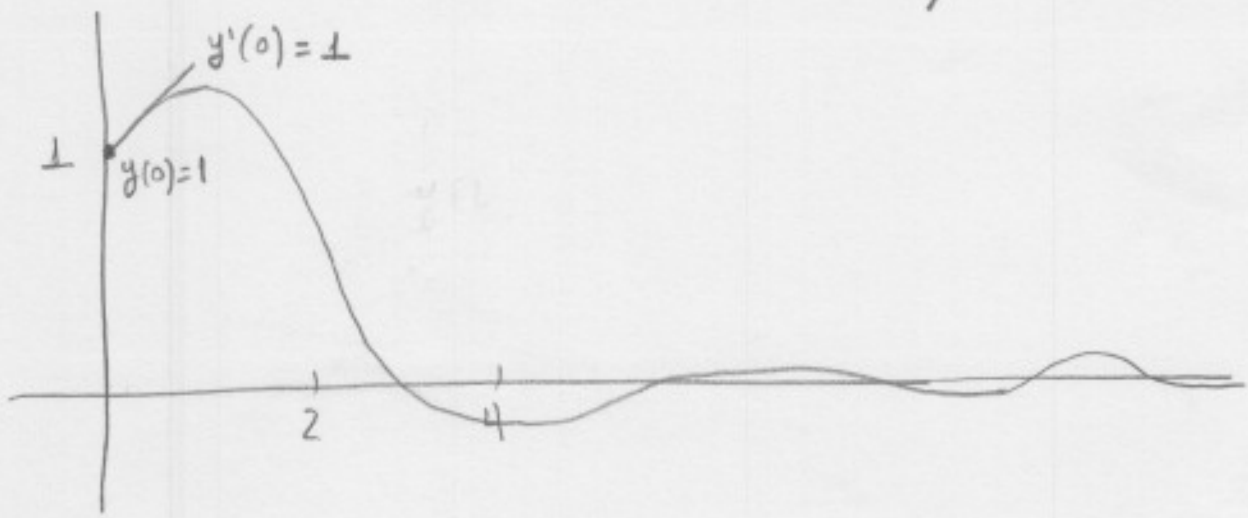
Note that $\lim_{t \to \infty} y(t) = 0$

To find the particular solution $y(0) = 1$, $y'(0) = 1$ we plug:

$$\left.\begin{aligned} y(0) &= c_1 = 1 \\ y'(0) &= -\frac{1}{2} c_1 + \frac{\sqrt{3}}{2} c_2 = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= \frac{3}{\sqrt{3}} = \sqrt{3} \end{aligned}$$

The particular solution is:

$$y(t) = e^{-\frac{t}{2}} \cos \frac{\sqrt{3} t}{2} + \sqrt{3}\, e^{-\frac{t}{2}} \sin \left(\frac{\sqrt{3} t}{2}\right)$$



Ex: $16y'' - 8y' + 145y = 0$

$y(0) = -2, \quad y'(0) = 1$

$16r^2 - 8r + 145 = 0$

$$r = \frac{8 \pm \sqrt{64 - 4(16)(145)}}{32}$$

$$r = \frac{8 \pm \sqrt{-9216}}{32} = \frac{8 \pm 96i}{32} = \frac{1}{4} \pm 3i$$

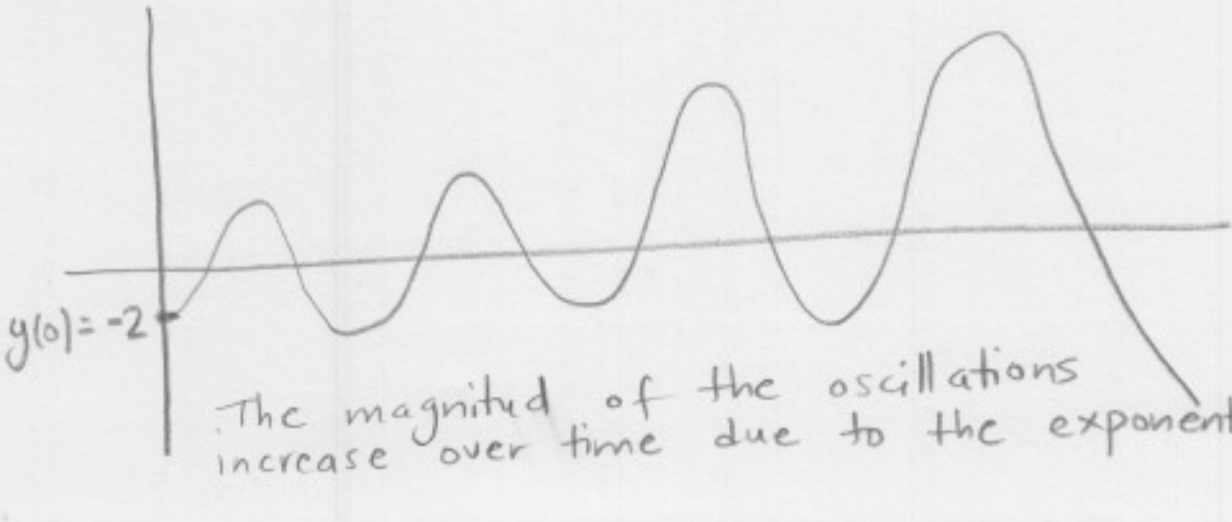
The general solution is:

$$y(t) = c_1 e^{\frac{t}{4}} \cos 3t + c_2 e^{\frac{t}{4}} \sin 3t$$

Imposing the initial conditions $y(0) = -2$, $y'(0) = 1$:

$$\left.\begin{aligned} y(0) &= c_1 = -2 \\ y'(0) &= \tfrac{1}{4} c_1 + 3c_2 = 1 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} c_1 &= -2 \\ c_2 &= \tfrac{1}{2} \end{aligned}$$

$$\Rightarrow \quad y(t) = -2e^{\frac{t}{4}} \cos 3t + \frac{1}{2} e^{\frac{t}{4}} \sin 3t$$



The magnitud of the oscillations increase over time due to the exponential $e^{\frac{t}{4}}$.