Section 3.4

Repeated roots; reduction of Order

Recall our second order linear homogeneous ODE:

$$ay'' + by' + cy = 0,$$

where a, b, c are constants.

Assuming an exponential solution leads to the characteristic equation:

$$y = e^{rt} \Rightarrow ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We have 3 cases:

Case 1: $r_1 \neq r_2$ real. We have solved this case:

$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$ is the general solution.

Case 2: $r_1 = \lambda + \mu i$, $r_2 = \lambda - \mu i$. We also solved this case:

$y(t) = c_1 e^{\lambda t} \cos \mu t + c_2 e^{\lambda t} \sin \mu t$ is the general solution.

Case 3: $r_1 = r_2 = -\frac{b}{2a}$ (in this case $b^2 - 4ac = 0$)

In this section we solve case 3.

We have one solution:

$$y_1(t) = e^{-\frac{b}{2a}t}$$

How do we find another solution $y_2(t)$?

We need $y_2(t)$ to be linearly independent with $y_1(t)$; that is, we look for:

$$y_2(t) = \nu(t)\, y_1(t) = \nu(t)\, e^{-\frac{b}{2a}t}$$

(if we choose $y_2(t) = c\, y_1(t)$, where c is just a number, then y_1 and y_2 are linearly dependent, and we would have $W(y_1, y_2) = 0$, which is not uselful: we can not apply the Wronskian theorem).

Therefore, $\nu(t)$ is a function that we need to find so that $y_2(t)$ solves $ay'' + by' + cy = 0$, case 3.

We have:

$$y_2(t) = \nu(t)\, e^{-\frac{b}{2a}t}$$

$$y_2'(t) = \nu'(t)\, e^{-\frac{b}{2a}t} - \frac{b}{2a}\nu(t)\, e^{-\frac{b}{2a}t}$$

$$y_2''(t) = \nu''\, e^{-\frac{b}{2a}t} - \frac{b}{2a}\nu'(t)\, e^{-\frac{b}{2a}t} - \frac{b}{2a}\nu'(t)\, e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}\nu(t)\, e^{-\frac{b}{2a}t}$$

We plug:

$$ay_2''(t) + by_2'(t) + cy_2(t) = a\left[\nu'' e^{-\frac{b}{2a}t} - \frac{b}{a}\nu' e^{-\frac{b}{2a}t} + \frac{b^2}{4a^2}\nu\, e^{-\frac{b}{2a}t}\right] + b\left[\nu' e^{-\frac{b}{2a}t} - \frac{b}{2a}\nu\, e^{-\frac{b}{2a}t}\right] + c\nu\, e^{-\frac{b}{2a}t}$$

$$= e^{-\frac{b}{2a}t}\left(a\nu'' - \cancel{b\nu'} + \frac{b^2}{4a}\nu + \cancel{b\nu'} - \frac{b^2}{2a}\nu + c\nu\right) \underset{\uparrow}{=} 0$$

want this equal to zero.

Since the exponential is never zero, we look for $v(t)$ so that:

$$av'' + \frac{b^2}{4a}v - \frac{b^2}{2a}v + cv = 0$$

or

$$av'' + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v = 0$$

$$av'' + \left(\frac{b^2 - 2b^2 + 4ac}{4a}\right)v = 0$$

$$av'' - \cancelto{0}{\frac{(b^2-4ac)}{4a}}\, v = 0$$

$$av'' = 0 \; ; \quad \text{since } b^2-4ac=0 \quad (\text{Case } 3)$$

Hence, integrating twice, we find:

$$v(t) = k_1 t + k_2$$

We choose $k_1 = 1$, $k_2 = 0$ $\Rightarrow$ $\boxed{v(t) = t}$

Hence:

$$\boxed{y_2(t) = t\, e^{-\frac{b}{2a}t}}$$

We compute $W(y_1,y_2)(t)$:

$$W(y_1,y_2)(t) = \begin{vmatrix} e^{-\frac{b}{2a}t} & t e^{-\frac{b}{2a}t} \\ -\frac{b}{2a} e^{-\frac{b}{2a}t} & e^{-\frac{b}{2a}t} - \frac{b}{2a} t\, e^{-\frac{b}{2a}t} \end{vmatrix}$$

$$W(y_1,y_2)(t) = e^{-\frac{b}{a}t} \cancel{-\frac{b}{2a}t}\, e^{-\frac{b}{a}t} \cancel{+\frac{b}{2a}t}\, e^{-\frac{b}{a}t}$$

$$= e^{-\frac{b}{a}t} \neq 0, \text{ for every } t.$$

Hence, the Wronskian theorem implies that $\left\{e^{-\frac{b}{2a}t},\ t e^{-\frac{b}{2a}t}\right\}$ form a fundamental set of solutions, and:

$$\boxed{y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}}$$

is the general solution to $ay'' + by' + cy = 0$, Case 3.

<u>Ex</u>: Consider the initial value problem:

$$\begin{cases} y'' + 2y' + y = 0 \\ y(0) = 1, \quad y'(0) = 1 \end{cases}$$

$$\Rightarrow \quad r^2 + 2r + 1 = 0$$

$$(r+1)^2 = 0 \quad \Rightarrow \quad r = -1 = -b/2a$$

One solution is $y_1(t) = e^{-t}$. We have shown that $y_2(t) = te^{-t}$ is another solution and that:

$$y(t) = c_1 e^{-t} + c_2 t e^{-t},$$

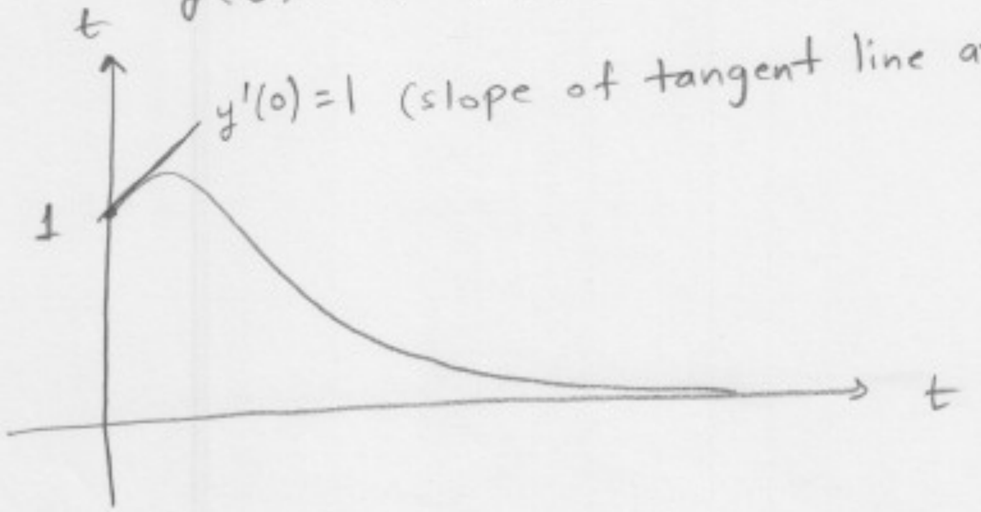
is the general solution to the equation.

We impose the initial conditions:

$$\left.\begin{aligned} y(0) &= c_1 = 1 \\ y'(0) &= -c_1 + c_2 = 1 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 1 \\ c_2 &= 2 \end{aligned}$$

Hence, the particular solution that passes through $(0,1)$ with slope $y'(0)=1$ is:

$$y(t) = e^{-t} + 2te^{-t}.$$



Note that $\lim_{t\to\infty} y(t) = 0$

Method of reduction of order.

We can extend the previous discussion to the second order, linear homogeneous equation:

$$y''(t) + p(t)y' + q(t)y = 0 \qquad (*)$$

Suppose we can find one solution to $(*)$ $y_1(t)$. We look for a second solution of the form:

$$\boxed{y_2(t) = v(t)\, y_1(t)}$$

We need to find $v(t)$ so that $y_2'' + py_2' + qy_2 = 0$.

$$y_2(t) = v(t) y_1(t)$$
$$y_2'(t) = v y_1' + y_1 v'$$
$$y_2''(t) = v y_1'' + v' y_1' + y_1 v'' + v' y_1'$$

We plug:

$$y_2'' + py_2' + qy_2 = y_1 v'' + 2v'y_1' + v y_1'' + p\,(v y_1' + y_1 v') + q v y_1$$
$$= y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v$$
$$= y_1 v'' + (2y_1' + py_1)v' + 0 \, ;$$

since y_1 is a solution, and thus $y_1'' + py_1' + qy_1 = 0$.

Hence, we need $v(t)$ so that:

$$\boxed{y_1 v'' + (2y_1' + py_1)v' = 0} \qquad (***)$$

Equation (***) is still second order, linear, but the term with v is not there.

Hence, we can perform a change:

Let $u(t) = v'(t)$, and hence $u'(t) = v''(t)$

Therefore, (***) becomes:

$$\boxed{y_1 u'(t) + (2y_1' + p y_1) u(t) = 0};$$

which is a <u>first order</u> linear equation for $u(t)$. Note that the coefficients $y_1(t)$ and $2y_1'(t) + p(t) y_1(t)$ are functions of t.

We can solve this first order equation using the method of integrating factor. Once we find $u(t)$, we find $v(t)$ by integrating,

$$v(t) = \int u(t)\,dt; \quad \text{since} \quad v'(t) = u(t).$$

Hence, we are able to find a second solution:

$$y_2(t) = v(t)\, y_1(t).$$

The method is called reduction of order because we ended up with first order equation (we reduced the problem from second order to first order).

Ex: Use the previous method to solve:

$$t^2 y'' + 3ty' + y = 0, \quad t>0, \quad y_1(t) = t^{-1}$$

We first double check that y_1 is a solution:

$$y_1(t) = t^{-1}$$
$$y_1'(t) = -t^{-2}$$
$$y_1''(t) = 2t^{-3}$$

We plug:

$$t^2 y_1'' + 3ty_1' + y_1 = t^2(2t^{-3}) + 3t(-t^{-2}) + t^{-1}$$
$$= 2t^{-1} - 3t^{-1} + t^{-1} = 0$$

Hence $y_1(t)$ is a solution

We look for $y_2(t)$ of the form:

$$y_2(t) = v(t)t^{-1}$$
$$y_2'(t) = v't^{-1} - v(t)t^{-2}$$
$$y_2'' = v''t^{-1} - v't^{-2} - v't^{-2} + 2vt^{-3}$$

Substituting and collecting terms:

$$t^2 y_2'' + 3ty_2' + y_2 = t^2(v''t^{-1} - 2v't^{-2} + 2vt^{-3})$$
$$+ 3t(v't^{-1} - vt^{-2}) + vt^{-1} = 0$$

We want $v(t)$ so that:

$$v''t - 2v' + \cancel{2vt^{-1}} + 3v' - \cancel{3vt^{-1}} + \cancel{vt^{-1}} = 0$$

$$\boxed{t\,v''(t) + v'(t) = 0}.$$

Letting:

$$u(t) = v'(t) \Rightarrow u'(t) = v''(t)$$

We have:

$$t u'(t) + u(t) = 0 \quad \text{or} \quad u'(t) + \frac{1}{t} u(t) = 0$$

We can use integrating factor or, in this case, separate variables:

$$\frac{du}{dt} = -\frac{u(t)}{t}$$

$$\int \frac{du}{u} = \int -\frac{dt}{t}$$

$$\ln|u| = -\ln|t| + C = -\ln t + C; \quad \text{since } t>0.$$

Hence:

$$|u| = c e^{-\ln t} = c e^{\ln t^{-1}} = c t^{-1}$$

$$\Rightarrow u = \pm c t^{-1} \quad \text{or} \quad u(t) = c t^{-1}$$

Choosing $c=1$ we find $u(t) = t^{-1}$, and hence:

$v(t) = \int \frac{1}{t} dt = \ln t + K$. We choose $K=0$.

We have found a second solution:

$$y_2(t) = v(t) y_1(t) = t^{-1} \ln t$$

We check that $W(y_1, y_2) \neq 0$.

$$W(y_1, y_2)(t) = \begin{vmatrix} t^{-1} & t^{-1}\ln t \\ -t^{-2} & \frac{t^{-1}}{t} - t^{-2}\ln t \end{vmatrix} = \frac{1}{t^3} \cancel{- \frac{1}{t^3}\ln t + \frac{1}{t^3}\ln t}$$
$$= \frac{1}{t^3} \neq 0, \text{ in } (0,\infty)$$

With $I = (0,\infty)$, we conclude:

$\boxed{y(t) = c_1 t^{-1} + c_2 t^{-1} \ln t \text{ is the general solution}}$