Section 3.5

Non-homogeneous equations; method of undetermined coefficients.

Recall the non-homogeneous equation:

$$y'' + p(t)y' + q(t)y = g(t), \quad (**)$$

where p, q, g are continuous functions on an open interval I.

The associated homogeneous equation is:

$$y'' + p(t)y' + q(t)y = 0 \quad (*)$$

We have the following:

<u>Theorem</u>: The general solution of the non-homogeneous equation:

$$y'' + p(t)y' + q(t)y = g(t) \quad (**)$$

can be written in the form:

$$y(t) = c_1 y_1 + c_2 y_2 + Y(t),$$

where y_1, y_2 form a fundamental set of solutions (i.e., $W(y_1, y_2)(t) \neq 0$ on I) of the homogeneous equation:

$$y'' + p(t)y' + q(t)y = 0 \quad (*),$$

c_1, c_2 are arbitrary constants and Y is a specific solution to the non-homogeneous equation $(**)$.

Proof:

Since Y is a particular solution of $(**)$ we have:

$$Y''(t)+p(t)Y'(t)+q(t)Y(t)=g(t).$$

Let $\tilde{Y}$ be any other solution of $(**)$. We consider the function $\tilde{Y}-Y$ and note that:

$$(\tilde{Y}-Y)''+p(t)(\tilde{Y}-Y)'+q(t)(\tilde{Y}-Y)=0 \qquad (***)$$

Indeed, the left side is:

$$\tilde{Y}''-Y''+p(t)(\tilde{Y}'-Y')+q(t)(\tilde{Y}-Y)=$$

$$\left(\tilde{Y}''+p(t)\tilde{Y}'+q(t)\tilde{Y}\right)-\left(Y''+p(t)Y'+q(t)Y\right)$$

$$=g(t)-g(t)=0,$$

since both $\tilde{Y}$ and Y are solutions to the non-homogeneous equation $(**)$.

From $(***)$ it follows that $\tilde{Y}-Y$ is a solution to the homogeneous equation $y''+py'+qy=0$. Since every solution of the homogeneous equation $(*)$ is of the form $c_1y_1+c_2y_2$, it follows that there exist constants α,β such that:

$$\tilde{Y}-Y=\alpha y_1+\beta y_2,$$

That is:
$$\tilde{Y}(t) = \bar{Y}(t) + \alpha y_1 + \beta y_2.$$

Since in this argument $\tilde{Y}$ is an arbitrary solution to (**) we have shown that every solution of (**) is contained in the family:
$$y(t) = c_1 y_1 + c_2 y_2 + \bar{Y}(t).$$
Conversely, every function in this family solves (**), so <u>the</u> general solution of (**) is
$$y(t) = c_1 y_1 + c_2 y_2 + \bar{Y}(t). \blacksquare$$

<u>Remark</u>: This theorem says that the general solution of (**) is equal to the solution to the homogeneous equation (*), plus a particular solution of (**); that is:
$$y(t) = y_H(t) + \bar{Y}(t),$$
where $y_H(t) = c_1 y_1 + c_2 y_2$ is the general solution to (*) and $\bar{Y}(t)$ solves (**). In general, it is hard to find $y_H(t)$, but we know how to do it if p, q are constants. Therefore we consider in this section the equation:
$$ay'' + by' + cy = g(t),$$
where $g(t)$ is a polynomial, exponential, sin, cos (or combinations of these function

Example 1: Exponential $g(t)$

- Consider the nonhomogeneous equation

$$\boxed{y'' - 3y' - 4y = 3e^{2t}}$$

$r^2 - 3r - 4 = 0$
$(r-4)(r+1) = 0$

$r_1 = 4 \quad r_2 = -1$

$Y(t) = c_1 e^{4t} + c_2 e^{-t}$

- We seek Y satisfying this equation. Since exponentials replicate through differentiation, a good start for Y is:

$$Y(t) = Ae^{2t} \Rightarrow Y'(t) = 2Ae^{2t},\ Y''(t) = 4Ae^{2t}$$

- Substituting these derivatives into differential equation,

$$4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 3e^{2t}$$

$$\Leftrightarrow -6Ae^{2t} = 3e^{2t} \quad \Leftrightarrow \quad A = -1/2$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = -\frac{1}{2}e^{2t}$$

$r^2 - 3r - 4 = 0$
$(r-4)(r+1) = 0$
$r_1 = 4 \quad r_2 = -1$

$\boxed{y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{2} e^{2t}}$ The general solution of equation $y'' - 3y' - 4y = 3e^{2t}$

Example 3: Polynomial $g(t)$

- Consider the nonhomogeneous equation

$$\boxed{y'' - 3y' - 4y = 4t^2 - 1}$$

- We seek Y satisfying this equation. We begin with

$$Y(t) = At^2 + Bt + C \Rightarrow Y'(t) = 2At + B,\ Y''(t) = 2A$$

- Substituting these derivatives into differential equation,

$$2A - 3(2At + B) - 4(At^2 + Bt + C) = 4t^2 - 1$$
$$\Leftrightarrow -4At^2 - (6A + 4B)t + (2A - 3B - 4C) = 4t^2 - 1$$
$$\Leftrightarrow -4A = 4,\ 6A + 4B = 0,\ 2A - 3B - 4C = -1$$
$$\Leftrightarrow A = -1,\ B = 3/2,\ C = -11/8$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = -t^2 + \frac{3}{2}t - \frac{11}{8}$$

General Solution $y(t) = c_1 e^{4t} + c_2 e^{-t} - t^2 + \frac{3}{2}t - \frac{11}{8} = c_1 y_1 + c_2 y_2 + Y(t)$.

Example 2: Sine $g(t)$, First Attempt (1 of 2)

- Consider the nonhomogeneous equation

$$\boxed{y'' - 3y' - 4y = 2\sin t}$$

- We seek Y satisfying this equation. Since sines replicate through differentiation, a good start for Y is:

$$Y(t) = A\sin t \Rightarrow Y'(t) = A\cos t,\ Y''(t) = -A\sin t$$

- Substituting these derivatives into differential equation,

$$-A\sin t - 3A\cos t - 4A\sin t = 2\sin t + 0\cos t$$

$$\Leftrightarrow (2 + 5A)\sin t + 3A\cos t = 0$$

$$\Leftrightarrow c_1 \sin t + c_2 \cos t = 0$$

- Since $\sin(x)$ and $\cos(x)$ are linearly independent (they are not multiples of each other), we must have $c_1 = c_2 = 0$, and hence $2 + 5A = 3A = 0$, which is impossible.

$$y'' - 3y' - 4y = 2\sin t$$

Example 2: Sine $g(t)$, Particular Solution (2 of 2)

- Our ~~next~~ attempt at finding a Y is

$$Y(t) = A\sin t + B\cos t$$
$$\Rightarrow Y'(t) = A\cos t - B\sin t,\ Y''(t) = -A\sin t - B\cos t$$

- Substituting these derivatives into ODE, we obtain

$$(-A\sin t - B\cos t) - 3(A\cos t - B\sin t) - 4(A\sin t + B\cos t) = 2\sin t$$
$$\Leftrightarrow (-5A + 3B)\sin t + (-3A - 5B)\cos t = 2\sin t \ + 0\cos t$$
$$\Leftrightarrow -5A + 3B = 2,\ -3A - 5B = 0$$
$$\Leftrightarrow A = -5/17,\ B = 3/17$$

- Thus a particular solution to the nonhomogeneous ODE is

$$Y(t) = \frac{-5}{17}\sin t + \frac{3}{17}\cos t$$

So $y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17}\sin t + \frac{3}{17}\cos t$ is the general solution

Example 4: Product $g(t)$

※ Consider the nonhomogeneous equation

$$\boxed{y'' - 3y' - 4y = -8e^t \cos 2t}$$

$Y(t) = (Ae^t)(\ \cos 2t + \ \sin 2t)$

※ We seek Y satisfying this equation, as follows:

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t$$

$$Y'(t) = Ae^t \cos 2t - 2Ae^t \sin 2t + Be^t \sin 2t + 2Be^t \cos 2t$$

$$= (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t$$

$$Y''(t) = (A + 2B)e^t \cos 2t - 2(A + 2B)e^t \sin 2t + (-2A + B)e^t \sin 2t$$

$$+ 2(-2A + B)e^t \cos 2t$$

$$= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t$$

※ Substituting derivatives into ODE and solving for A and B:

$$A = \frac{10}{13},\ B = \frac{2}{13} \quad \Rightarrow Y(t) = \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t$$

$\Rightarrow \ y(t) = c_1 e^{4t} + c_2 e^{-t} + \frac{10}{13} e^t \cos 2t + \frac{2}{13} e^t \sin 2t$ General solution

Discussion: Sum $g(t)$

- Consider again our general nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

- Suppose that $g(t)$ is sum of functions:

$$g(t) = g_1(t) + g_2(t)$$

- If Y_1, Y_2 are solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$
$$y'' + p(t)y' + q(t)y = g_2(t)$$

respectively, then $Y_1 + Y_2$ is a solution of the nonhomogeneous equation above.

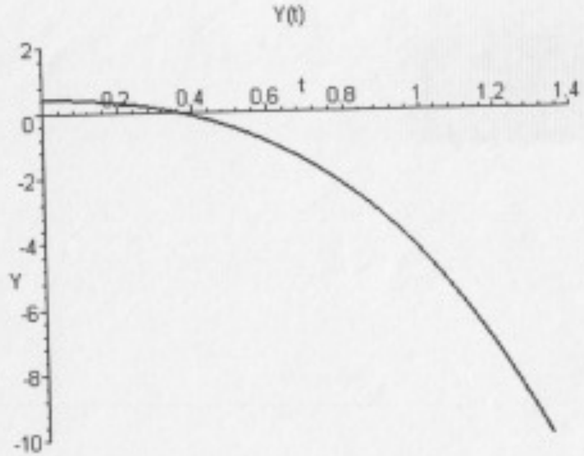
Example 5: Sum $g(t)$

- Consider the equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^{t}\cos 2t$$

- Our equations to solve individually are

$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^{t}\cos 2t$$



- Our particular solution is then

$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^{t}\cos 2t + \frac{2}{13}e^{t}\sin 2t$$

Example 6: First Attempt (1 of 3)

- Consider the equation

$$y'' + 4y = 3\cos 2t$$

- We seek Y satisfying this equation. We begin with

$$Y(t) = A\sin 2t + B\cos 2t$$

$$\Rightarrow Y'(t) = 2A\cos 2t - 2B\sin 2t,\ Y''(t) = -4A\sin 2t - 4B\cos 2t$$

- Substituting these derivatives into ODE:

$$\begin{aligned}(-4A\sin 2t - 4B\cos 2t) + 4(A\sin 2t + B\cos 2t) &= 3\cos 2t \\ (-4A + 4A)\sin 2t + (-4B + 4B)\cos 2t &= 3\cos 2t \\ 0 &= 3\cos 2t\end{aligned}$$

- Thus no particular solution exists of the form

$$Y(t) = A\sin 2t + B\cos 2t$$

Example 6: Homogeneous Solution (2 of 3)

- Thus no particular solution exists of the form

$$Y(t) = A\sin 2t + B\cos 2t$$

- To help understand why, recall that we found the corresponding homogeneous solution

$$y'' + 4y = 0 \;\Rightarrow\; y(t) = c_1\cos 2t + c_2\sin 2t$$

- Thus our assumed particular solution solves homogeneous equation

$$y'' + 4y = 0$$

instead of the nonhomogeneous equation.

$$y'' + 4y = 3\cos 2t$$

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$$y'' + 4y = 3\cos 2t$$

Example 6: Particular Solution (3 of 3)

- Our next attempt at finding a Y is:

$$Y(t) = At\sin 2t + Bt\cos 2t$$

$$Y'(t) = A\sin 2t + 2At\cos 2t + B\cos 2t - 2Bt\sin 2t$$

$$Y''(t) = 2A\cos 2t + 2A\cos 2t - 4At\sin 2t - 2B\sin 2t - 2B\sin 2t - 4Bt\cos 2t$$

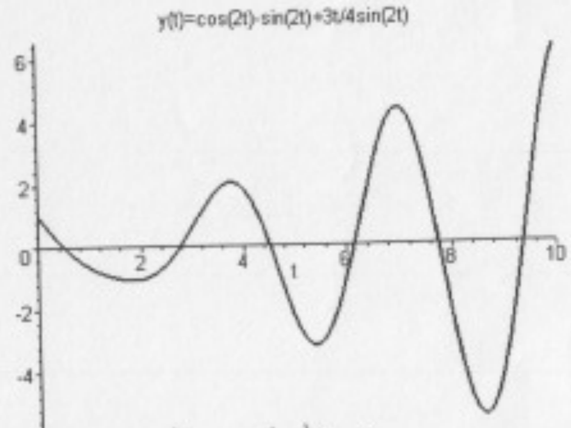
$$= 4A\cos 2t - 4B\sin 2t - 4At\sin 2t - 4Bt\cos 2t$$

- Substituting derivatives into ODE,

$$4A\cos 2t - 4B\sin 2t = 3\cos 2t$$

$$\Rightarrow A = 3/4,\ B = 0$$

$$\Rightarrow Y(t) = \frac{3}{4} t\sin 2t$$



$y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{3}{4} t \sin 2t$ general solution