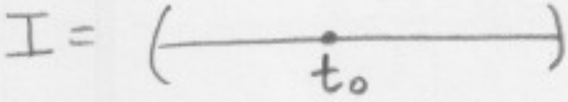
Section 3.6
Variation of parameters

Recall the non-homogeneous equation:

$$y'' + p(t)y' + q(t)y = g(t), \qquad (**)$$

where $p(t)$, $q(t)$, $g(t)$ are continuous functions on an open interval I



The associated homogeneous equation is:

$$y'' + p(t)y' + q(t)y = 0. \qquad (*).$$

The method of undetermined coefficients works for the case when p, q are constants and $g(t)$ is an exponential, polynomial, sine, cosine, or combinations of these functions.

Recall that the general solution of $(**)$ is:

$$y(t) = c_1 y_1 + c_2 y_2 + Y(t),$$

where $y_H(t) = c_1 y_1 + c_2 y_2$ is the general solution of $(*)$, that is, $\{y_1, y_2\}$ form a fundamental set of solutions of $(*)$ (hence $W(y_1, y_2)(t) \neq 0$) and $Y(t)$ is a particular solution of $(**)$:

$$Y'' + p(t)Y' + q(t)Y = g(t).$$

The method of variation of parameters allows for more general right hand side $g(t)$, but requires that we have a fundamental set of solutions $\{y_1, y_2\}$ in our hand.

In this case, we look for a solution:

$$Y(t) = u_1(t) y_1(t) + u_2(t) y_2(t),$$

where $u_1(t), u_2(t)$ need to be found so that $Y(t)$ solves:

$$Y'' + p(t) Y' + q(t) Y = g(t)$$

We have:

$$Y'(t) = u_1 y_1' + y_1 u_1' + u_2 y_2' + y_2 u_2'$$

We impose the condition:

$$\boxed{y_1 u_1' + y_2 u_2' = 0} \quad (1)$$

With (1), $Y'(t)$ simplifies to:

$$Y'(t) = u_1 y_1' + u_2 y_2'$$

Hence

$$Y''(t) = u_1 y_1'' + y_1' u_1' + u_2 y_2'' + y_2' u_2'$$

We now plug in (**):

$$Y''(t) + p(t)Y'(t) + q(t)Y(t) = g(t)$$

$\uparrow$ want.

$$(u_1 y_1'' + y_1' u_1' + u_2 y_2'' + y_2' u_2') + p(t)\,(u_1 y_1' + u_2 y_2')$$
$$+ q(t)\,(u_1 y_1 + u_2 y_2) = g(t)$$

We collect terms on the left:

$$u_1 y_1'' + p(t) u_1 y_1' + q(t) u_1 y_1 + u_2 y_2'' + p(t) u_2 y_2' + q u_2 y_2$$
$$+ y_1' u_1' + y_2' u_2' = g(t)$$

$$u_1 (y_1'' + p(t) y_1' + q(t) y_1) + u_2 (y_2'' + p(t) y_2' + q y_2)$$
$$+ y_1' u_1' + y_2' u_2' = g(t) \qquad (2)$$

Since y_1, y_2 are solutions to the homogeneos equation we have:

$$y_1'' + p(t) y_1' + q(t) y_1 = 0, \quad y_2'' + p(t) y_2' + q(t) y_2 = 0$$

Therefore, (2) simplifies to:

$$\boxed{y_1' u_1' + y_2' u_2' = g(t)} \quad (3)$$

From (1) and (3) we obtain the 2x2 system (for each t):

$$\boxed{\begin{aligned} y_1(t)\,u_1'(t) + y_2(t)\,u_2'(t) &= 0 \\ y_1'(t)\,u_1'(t) + y_2'(t)\,u_2'(t) &= g(t) \end{aligned}}$$

This is a system of the form:

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= e, \end{aligned}$$

where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix}$

and $x = u_1'(t)$, $y = u_2'(t)$, $e = g(t)$

Since $\{y_1, y_2\}$ is a fundamental set of solutions we have that $W(y_1,y_2)(t) \neq 0$; that is:

$$\det \begin{pmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{pmatrix} \neq 0.$$

Since A is invertible the system:

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ e \end{pmatrix}$$

has a unique solution:

$$A^{-1} A \begin{pmatrix} x \\ y \end{pmatrix} = A^{-1} \begin{pmatrix} 0 \\ e \end{pmatrix}$$

Since $A^{-1}A =$ Identity matrix $= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

we obtain:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ e \end{pmatrix}.$$

Ex: Find the general solution of:

$$t^2 y'' - 2y = 3t^2 - 1, \quad t > 0, \quad I = (0, \infty)$$

where $\{y_1, y_2\} = \{t^2, t^{-1}\}$ is a fundamental set of solutions.

We first check that $\{t^2, t^{-1}\}$ is a fundamental set of solutions:

$y_1(t) = t^2$, $y_1' = 2t$, $y_1'' = 2$, plug:

$$t^2(2) - 2(t^2) = 0$$
$$0 = 0 \ \checkmark$$

$y_2(t) = t^{-1}$ $\quad y_2' = -t^{-2}$, $y_2'' = 2t^{-3}$, plug:

$$t^2(2t^{-3}) - 2(t^{-1}) = 0$$
$$2t^{-1} - 2t^{-1} = 0$$
$$0 = 0 \ \checkmark$$

Check that $W(y_1, y_2)(t) \neq 0$.

$$\det \begin{pmatrix} t^2 & t^{-1} \\ 2t & -t^{-2} \end{pmatrix} = -t^0 - 2t^0 = -3 \neq 0$$

Hence $y_H(t) = c_1 t^2 + c_2 t^{-1}$ is the general solution of the homogeneous equation $t^2 y'' - 2y = 0$.

We look for:

$$Y(t) = u_1 y_1 + u_2 y_2$$

Our equation is:

$$y'' - \frac{2}{t^2} y = 3 - \frac{1}{t^2}$$

$p(t) = 0$ $\quad q(t) = -\frac{2}{t^2}$, $\quad g(t) = 3 - \frac{1}{t^2}$, all continuous functions in $(0, \infty)$.

From previous discussion, we need to solve the following 2×2 system:

$$y_1(t)\, u_1'(t) + y_2(t)\, u_2'(t) = 0$$
$$y_1'(t)\, u_1'(t) + y_2'(t)\, u_2'(t) = g(t)$$

$$\Rightarrow \quad t^2 u_1'(t) + t^{-1} u_2'(t) = 0$$
$$2t\, u_1'(t) - t^{-2} u_2'(t) = 3 - \frac{1}{t^2}$$

$$\Rightarrow u_2'(t) = -\frac{t^2 u_1'(t)}{t^{-1}} = -t^3 u_1'(t)$$

$$\Rightarrow 2t\, u_1'(t) - t^{-2}\left(-t^3 u_1'(t)\right) = 3 - \frac{1}{t^2}$$
$$3t\, u_1'(t) = 3 - \frac{1}{t^2}$$
$$\Rightarrow \boxed{u_1'(t) = \frac{1}{t} - \frac{1}{3t^3}}$$

$$\Rightarrow \quad u_2'(t) = -t^3\left(\frac{1}{t} - \frac{1}{3t^3}\right) = -t^2 + \frac{1}{3}$$
$$\boxed{u_2'(t) = \frac{1}{3} - t^2}$$

We found:

$$u_1'(t) = \frac{1}{t} - \frac{1}{3t^3} \Rightarrow u_1(t) = \int\left(\frac{1}{t} - \frac{1}{3}t^{-3}\right)dt$$

$$\boxed{u_1(t) = \ln t + \frac{1}{6t^2}}$$

Choose constant of integration equal to zero

$$u_2'(t) = \frac{1}{3} - t^2 \Rightarrow u_2(t) = \int\left(\frac{1}{3} - t^2\right)dt$$

$$\boxed{u_2(t) = \frac{t}{3} - \frac{t^3}{3}}$$

Hence:

$$\begin{aligned} Y(t) &= u_1(t)\,y_1(t) + u_2(t)\,y_2(t) \\ &= \left(\ln t + \frac{1}{6t^2}\right)t^2 + \left(\frac{t}{3} - \frac{t^3}{3}\right)t^{-1} \\ &= t^2\ln t + \frac{1}{6} + \frac{1}{3} - \frac{t^2}{3} \\ &= t^2\ln t - \frac{t^2}{3} + \frac{1}{2} \end{aligned}$$

The general solution of $t^2y'' - 2y = 3t^2 - 1$, $t > 0$, is:

$$y(t) = y_H(t) + Y(t)$$

$$y(t) = c_1t^2 + c_2t^{-1} + \left(\frac{1}{2} + t^2\ln t - \frac{t^2}{3}\right)$$

$$\boxed{y(t) = c_1t^2 + c_2t^{-1} + \left(t^2\ln t + \frac{1}{2}\right);}$$

since the term $-t^2/3$ can be absorbed in the first term.

Double check: $Y(t) = t^2\ln t + \frac{1}{2}$ solves $t^2y'' - 2y = 3t^2 - 1$

$Y'(t) = 2t\ln t + t$, $Y'' = 2\ln t + 2 + 1 = 3 + 2\ln t$

$$t^2Y'' - 2Y = t^2(3 + 2\ln t) - 2\left(t^2\ln t + \frac{1}{2}\right) = 3t^2 + \cancel{2t^2\ln t} - \cancel{2t^2\ln t} - 1 = 3t^2 - 1.$$ ✓