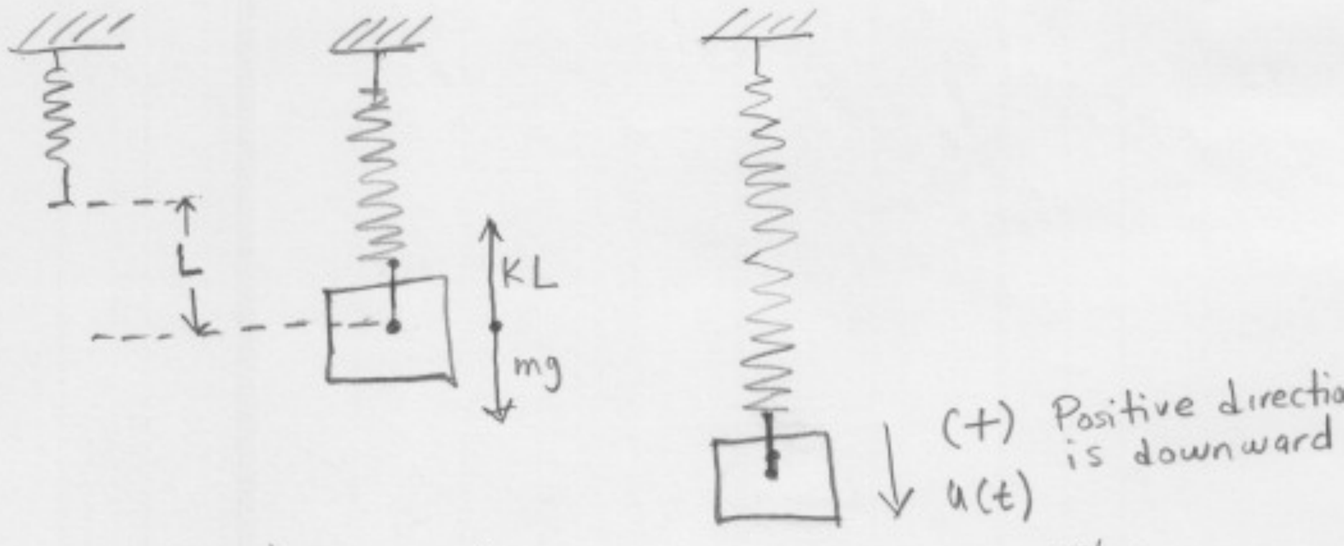
Section 3.7

Mechanical and electrical vibrations.

Two important areas of application for second order linear equations with constant coefficients are:

(1) Modeling of mechanical oscillations
(2) Modeling of electrical oscillations

For (1) we will study the motion of a mass on a spring. An understanding of the behavior of this simple system is the first step in investigation of more complex vibrating systems.



A mass m hangs from vertical spring. The mass causes an elongation L of the spring.

- The force F_G of gravity pulls mass down. This force has magnitud mg, where g is acceleration due to gravity.

- The force F_S of spring stiffness pulls mass up. This force is proportional to displacement (Hooke's Law).

Before starting the motion, when the mass is in equilibrium, the two forces F_G and F_S balance each other:

(Hooke's law)
$F_S = KL$

$F_G = mg$

$$\boxed{mg = KL}$$

The motion of the mass starts when it is acted on by an external force (forcing function) or is initially displaced.

Let $u(t)$ denote the displacement of the mass from its equilibrium position at time t, measured downward.

Let $f(t)$ be the net force acting on mass. The motion is governed by Newton's second law:

$$m u''(t) = f(t).$$

$f(t)$ is composed of 4 different forces;

1.- Weight $w = mg$ (downward force) $\downarrow (+)$

2.- Spring force: $F_S(t) = -K(L + u(t))$ Hooke's law

3.- Damping force: $F_d(t) = -\gamma u'(t)$

The damping force is proportional to the velocity and it acts trying to stop the motion

4.- External force: $F(t)$

Hence

$$
\begin{aligned}
m u''(t) &= f(t) \\
&= mg + F_S(t) + F_d(t) + F(t) \\
&= mg - K(L + u(t)) - \gamma u'(t) + F(t) \\
&= \underbrace{mg - KL}_{\text{zero}} - K u(t) - \gamma u'(t) + F(t) \\
&= -K u(t) - \gamma u'(t) + F(t)
\end{aligned}
$$

Hence;

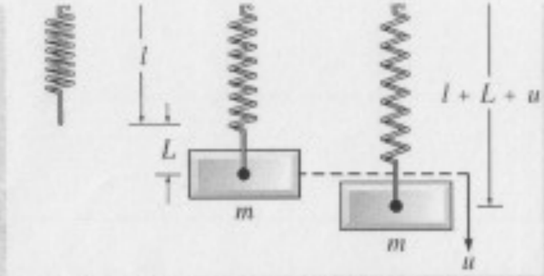
$$m u''(t) + \gamma u'(t) + K u(t) = F(t)$$

We must have two initial conditions, the initial displacement $u(0)$ and the initial velocity $u'(0)$. Therefore, we have found that the spring is modeled with the following initial value problem:

$$\text{IVP}\begin{cases} mu''(t)+\gamma u'(t)+ku(t)=F(t) \\ u(0)=u_0, \quad u'(0)=\nu_0 \end{cases}$$

Since m, γ, k are constants, and if $F(t)$ is a continuous function, the existence theorem guarantees a unique solution to IVP. Physically, if the mass is set in motion with a given initial displacement and velocity, then its position is uniquely determined at all future times. The position is given by the function $u(t)$.

Example 1:
Find Coefficients (1 of 2)



- A 4 lb mass stretches a spring 2". The mass is displaced an additional 6" and then released; and is in a medium that exerts a viscous resistance of 6 lb when velocity of mass is 3 ft/sec. Formulate the IVP that governs motion of this mass:

$$\boxed{mu''(t)+\gamma u'(t)+ku(t)=F(t),\ u(0)=u_0,\ u'(0)=v_0}$$

- Find m:

$$w=mg \Rightarrow m=\frac{w}{g} \Rightarrow m=\frac{4\,\text{lb}}{32\text{ft}/\sec^2} \Rightarrow m=\frac{1}{8}\frac{\text{lb}\sec^2}{\text{ft}}$$

- Find γ :

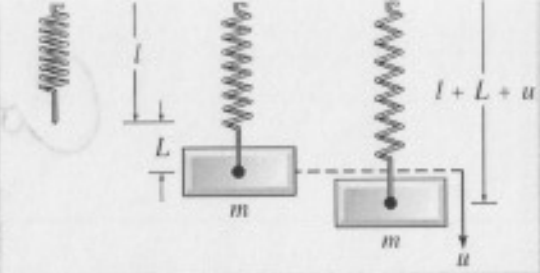
$$\gamma u'=6\text{lb} \Rightarrow \gamma=\frac{6\,\text{lb}}{3\text{ft}/\sec} \Rightarrow \gamma=2\frac{\text{lb}\sec}{\text{ft}}$$

- Find k:

$$F_s=-k\,L \Rightarrow k=\frac{4\,\text{lb}}{2\,\text{in}} \Rightarrow\ k=\frac{4\,\text{lb}}{1/6\,\text{ft}} \Rightarrow k=24\frac{\text{lb}}{\text{ft}}$$

4 = KL

Example 1: Find IVP (2 of 2)



- Thus our differential equation becomes

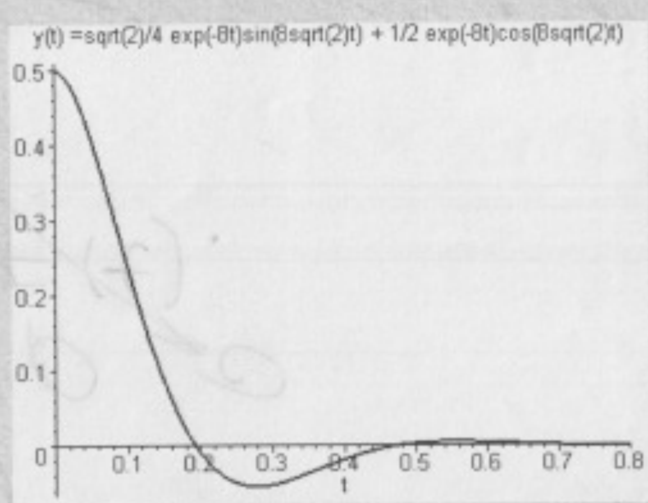
$$\frac{1}{8}u''(t)+2u'(t)+24u(t)=0$$

and hence the initial value problem can be written as

$$u''(t)+16u'(t)+192u(t)=0$$

$$u(0)=\frac{1}{2},\quad u'(0)=0$$

- This problem can be solved using methods of Chapter Section 3.4. Given on right is the graph of solution.



$r^2+16r+192=0$

$r=\frac{-16\pm\sqrt{16^2-4\cdot 192}}{2}$

$4\overline{)192}$ 48, 32

$4\cdot 4\cdot 16\cdot 3$

$16^2\cdot 3$

$r=\frac{-16\pm 16\sqrt{1-3}}{2}=-8\pm 8\sqrt{2}i$

$u(t)=c_1e^{-8t}\cos 8\sqrt{2}t+c_2e^{-8t}\sin 8\sqrt{2}t$

$u(0)=c_1=1/2$

$u'(t)=-8c_1e^{-8t}\cos 8\sqrt{2}t+\ldots+8\sqrt{2}c_2e^{-8t}\cos 8\sqrt{2}t$

$u'(0)=-8c_1+8\sqrt{2}c_2=$

$\sqrt{2}c_2=\frac{1}{2}$

$c_2=\frac{\sqrt{2}}{4}$

Spring Model: Undamped Free Vibrations (1 of 4)

- Recall our differential equation for spring motion:

$$mu''(t) + \gamma u'(t) + ku(t) = F(t)$$

- Suppose there is no external driving force and no damping. Then $F(t) = 0$ and $\gamma = 0$, and our equation becomes

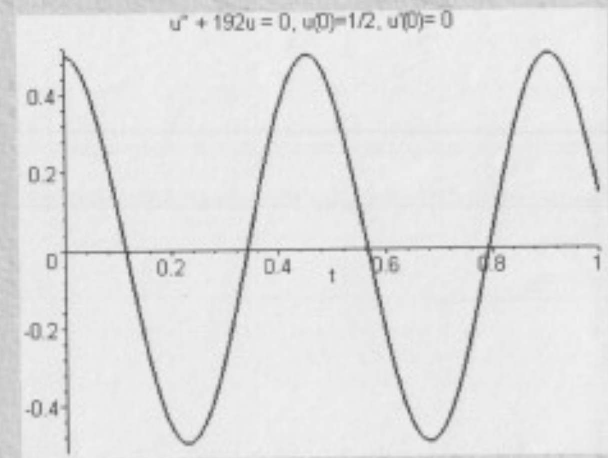
$$mu''(t) + ku(t) = 0$$

- The general solution to this equation is

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t,$$

where

$$\omega_0^2 = k/m$$



Spring Model: Undamped Free Vibrations (2 of 4)

- Using trigonometric identities, the solution

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t, \quad \omega_0^2 = k/m$$

can be rewritten as follows:

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \iff u(t) = R\cos(\omega_0 t - \delta)$$

$$\iff u(t) = \underbrace{R\cos\delta}_{A}\cos\omega_0 t + \underbrace{R\sin\delta}_{B}\sin\omega_0 t,$$

where

$$A = R\cos\delta, \; B = R\sin\delta \;\Rightarrow\; R = \sqrt{A^2 + B^2}, \; \tan\delta = \frac{B}{A}$$

- Note that in finding δ, we must be careful to choose correct quadrant. This is done using the signs of $\cos\delta$ and $\sin\delta$.

Spring Model: Undamped Free Vibrations (3 of 4)

- Thus our solution is

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \;= R\cos(\omega_0 t - \delta)$$

where

$$\omega_0 = \sqrt{k/m}$$

- The solution is a shifted cosine (or sine) curve, that describes simple harmonic motion, with period

$$T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}}$$

- The circular frequency ω_0 (radians/time) is **natural frequency** of the vibration, R is the **amplitude** of max displacement of mass from equilibrium, and δ is the **phase** (dimensionless).

Example 2: Find IVP (1 of 3)

- A 10 lb mass stretches a spring 2". The mass is displaced an additional 2" and then set in motion with initial upward velocity of 1 ft/sec. Determine position of mass at any later time. Also find period, amplitude, and phase of the motion.

$$mu''(t)+ku(t)=0,\ u(0)=u_0,\ u'(0)=v_0$$

- Find m:

$$w=mg \Rightarrow m=\frac{w}{g} \Rightarrow m=\frac{10\,\text{lb}}{32\,\text{ft}/\text{sec}^2} \Rightarrow m=\frac{5}{16}\frac{\text{lb}\,\text{sec}^2}{\text{ft}}$$

- Find k:

$$F_s=-kL \Rightarrow k=\frac{10\,\text{lb}}{2\,\text{in}} \Rightarrow k=\frac{10\,\text{lb}}{1/6\,\text{ft}} \Rightarrow k=60\frac{\text{lb}}{\text{ft}}$$

- Thus our IVP is

$$5/16u''(t)+60u(t)=0,\ u(0)=1/6,\ u'(0)=-1$$

$$u(t)=\frac{1}{6}\cos 8\sqrt{3}t-\frac{1}{8\sqrt{3}}\sin 8\sqrt{3}t$$

$$u(t)=0.182\cos(8\sqrt{3}t+0.409)$$

Example 2: Find Solution (2 of 3)

- Simplifying, we obtain

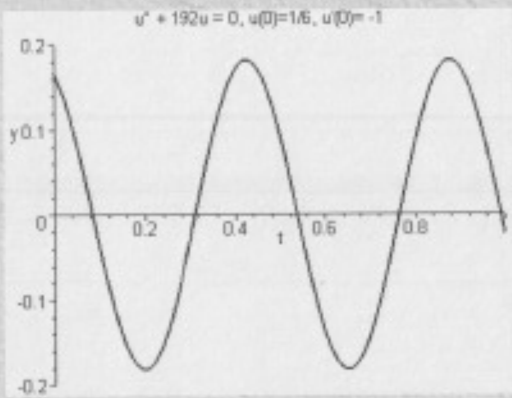
$$u''(t)+192u(t)=0,\quad u(0)=1/6,\quad u'(0)=-1$$

- To solve, use methods of Ch 3.4 to obtain

$$u(t)=\underbrace{\frac{1}{6}}_{A}\cos\sqrt{192}\,t-\underbrace{\frac{1}{\sqrt{192}}}_{B}\sin\sqrt{192}\,t \;=\; \underbrace{R\cos}_{T}(\omega_0 t + \delta)$$

or

$$u(t)=\frac{1}{6}\cos 8\sqrt{3}\,t-\frac{1}{8\sqrt{3}}\sin 8\sqrt{3}\,t$$



$$u(t)=\frac{1}{6}\cos 8\sqrt{3}t-\frac{1}{8\sqrt{3}}\sin 8\sqrt{3}t$$

Example 2: Find Period, Amplitude, Phase (3 of 3)

- The natural frequency is

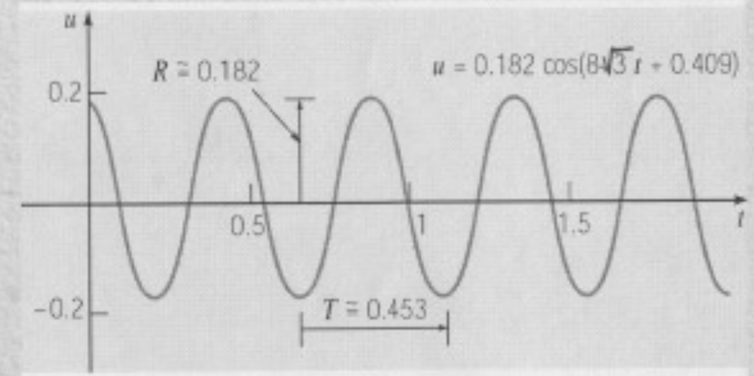
  $$\omega_0=\sqrt{k/m}=\sqrt{192}=8\sqrt{3}\cong 13.856 \text{ rad/sec}$$

- The period is

  $$T=2\pi/\omega_0\cong 0.45345 \text{ sec}$$

- The amplitude is

  $$R=\sqrt{A^2+B^2}\cong 0.18162 \text{ ft}$$



- Next, determine the phase δ :

  $$A=R\cos\delta,\ \ B=R\sin\delta,\ \ \tan\delta=B/A$$

  $$\tan\delta=\frac{B}{A}\Rightarrow \tan\delta=\frac{-\sqrt{3}}{4}\Rightarrow \delta=\tan^{-1}\left(\frac{-\sqrt{3}}{4}\right)\cong -0.40864 \text{ rad}$$

Thus $u(t)=0.182\cos\left(8\sqrt{3}t+0.409\right)$