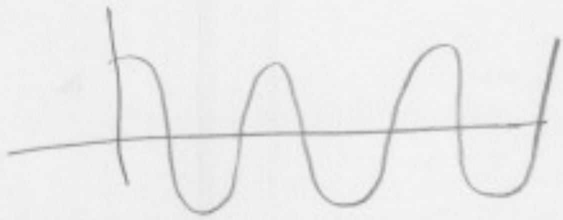
Section 3.8
Forced vibrations

Recall our model for the spring:

$$mu''(t) + \gamma u'(t) + Ku(t) = F(t)$$
$$u(0) = u_0, \quad u'(0) = v_0$$

If $\gamma = 0$ and $F(t) = 0$, from last class, the solution can be written as:

$$u(t) = R\cos(\omega_0 t - \delta)$$



$\gamma = 0$

The oscillation continues as $t \to \infty$. Since $\gamma = 0$, there is no damping force to stop the motion.

In reality, there is always some damping force, even if very small. In this section we study the effect of the damping coefficient γ on the system. Hence, we consider:

$$mu''(t) + \gamma u'(t) + Ku(t) = 0$$

The characteristic equation is:

$$mr^2 + \gamma r + K = 0$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mK}}{2m}$$

$$r = \frac{-\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4mK}}{2m}$$

$$r = \frac{-\gamma}{2m} \pm \frac{\gamma}{2m} \frac{\sqrt{\gamma^2 - 4mK}}{\gamma}$$

$\Rightarrow$

$$r = -\frac{\gamma}{2m} \pm \frac{\gamma}{2m}\sqrt{\frac{\gamma^2-4mK}{\gamma^2}}$$

$$r = \frac{\gamma}{2m}\left[-1 \pm \sqrt{1-\frac{4mK}{\gamma^2}}\right]$$

We have 3 cases:

Case 1: $\gamma^2-4mK>0$, or $\gamma > 2\sqrt{mK}$

In this case note that both r_1, r_2 are negative $\left(\text{since } 1-\frac{4mK}{\gamma^2} < 1\right)$.

Thus, the general solution is:

$$u(t) = Ae^{r_1 t} + Be^{r_2 t}$$

Case 2: $\gamma^2-4mK=0$, or $\gamma = 2\sqrt{mK}$.

In this case we have only one root: $r = \frac{-\gamma}{2m}$

The general solution is:

$$u(t) = Ae^{-\frac{\gamma}{2m}t} + Bt\,e^{-\frac{\gamma}{2m}t}$$

Case 3: $\gamma^2-4mK<0$, or $\gamma < 2\sqrt{mK}$

In this case, $r_1 = \lambda+\mu i$, $r_2 = \lambda-\mu i$, where $\lambda = \frac{-\gamma}{2m}$, $\mu = \frac{\sqrt{4mK-\gamma^2}}{2m}$. The general solution is:

$$u(t) = A\,e^{-\frac{\gamma}{2m}t}\cos\mu t + B\,e^{-\frac{\gamma}{2m}t}\sin\mu t.$$

<u>Remark</u>: In all 3 cases:

$$\lim_{t \to \infty} u(t) = 0.$$

The value $\gamma = 2\sqrt{km}$ is known as the <u>critical damping value</u>, and for larger values of γ the motion is said to be <u>overdamped</u>. Hence, the solution in case 1 is overdamped, and the solution in case 2 is critically damped.

We now examine Case 3:

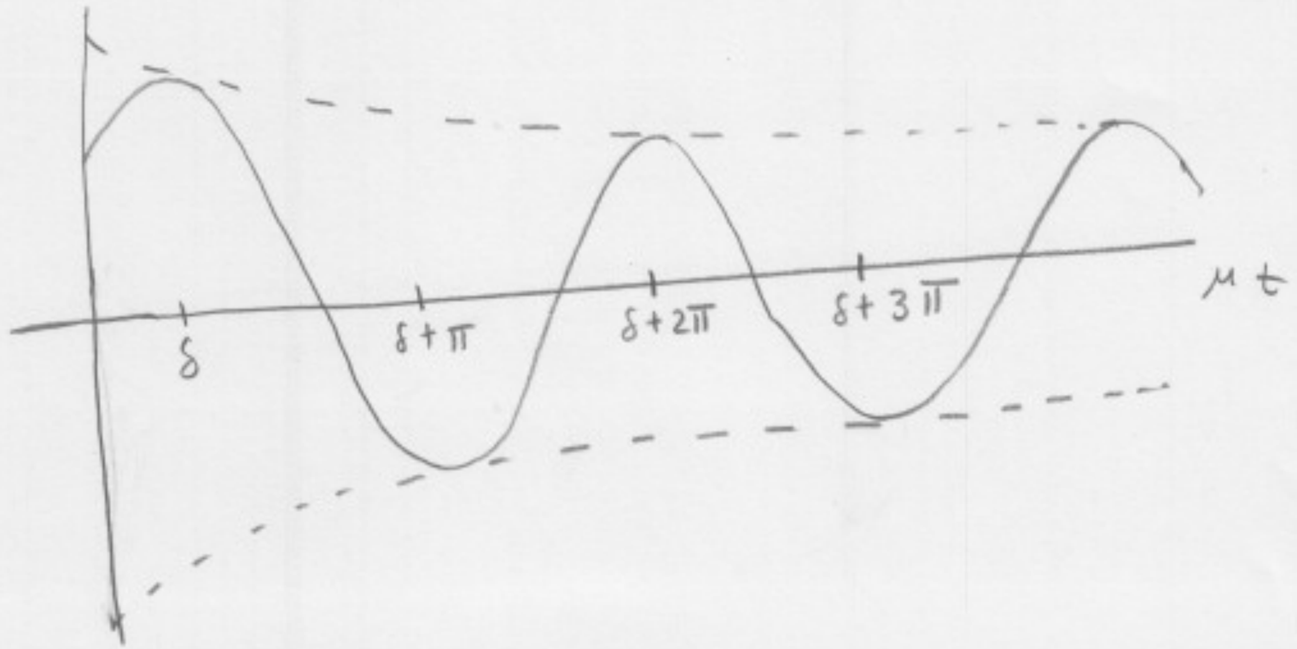
$$u(t) = e^{-\frac{\gamma t}{2m}} \left(A\cos \mu t + B \sin \mu t \right), \quad \mu > 0$$

We proceed as in the previous class and rewrite $u(t)$ as:

$$u(t) = e^{-\frac{\gamma}{2m} t} \left(R \cos (\mu t - \delta) \right)$$

Hence:

$$|u(t)| \leq R\, e^{-\frac{\gamma}{2m} t}$$



Note that in case 3, the solution is oscillating, but the oscillations get smaller due to the term $e^{-\frac{\gamma}{2m}t}$ which goes to zero as $t \to \infty$.

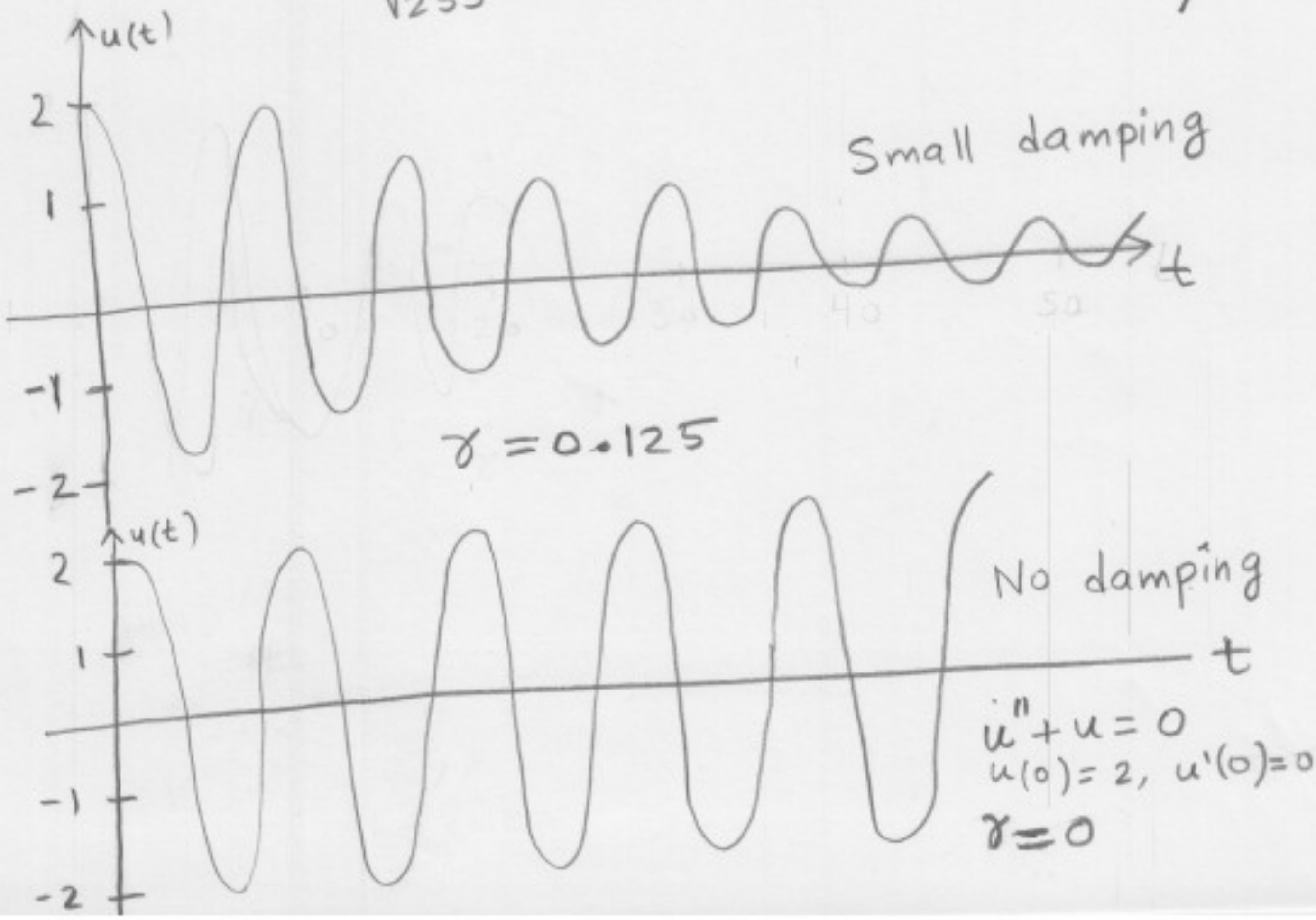
<u>Ex</u> : $u'' + 0.125u' + u = 0$
$u(0) = 2, \quad u'(0) = 0.$

At this point, we know how to solve this equation:

$$u(t) = e^{-\frac{t}{16}}\left(2\cos\frac{\sqrt{255}}{16}t + \frac{2}{\sqrt{255}}\sin\frac{\sqrt{255}}{16}t\right)$$

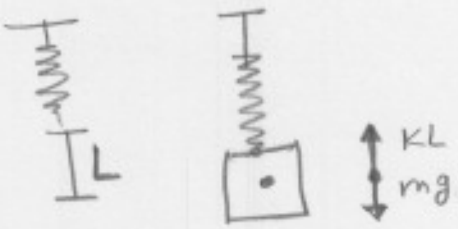
$$= e^{-\frac{t}{16}} R\cos\left(\frac{\sqrt{255}}{16}t - \delta\right)$$

$$= e^{-\frac{t}{16}} \frac{32}{\sqrt{255}}\cos\left(\frac{\sqrt{255}}{16}t - 0.06254\right)$$



Forced vibrations.

Ex. A mass weighing 4 lb stretches a spring 1.5 in. The mass is displaced 2 in in the possitive direction from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of 2 Cos 3t lb, formulate the IVP describing the motion of the mass.



$L = 1.5\text{ in} = \frac{1.5}{12}\text{ ft}$, $\quad \gamma = 0$ (no damping)

$$W = 4 = mg \Rightarrow m = \frac{4}{32} = \frac{1}{8}$$

Hooke's Law: $mg = KL$

$$K = \frac{mg}{L} = \frac{4}{\frac{1.5}{12}} = \frac{48}{1.5} = \frac{480}{15} = 32$$

$$\Rightarrow \quad mu'' + \gamma u' + Ku = F(t)$$

$$\frac{1}{8}u'' + 32u = 2\cos 3t$$

$$\underset{\text{IVP}}{\Rightarrow} \begin{cases} u''(t) + 256\,u(t) = 16\cos 3t \\ u(0) = \frac{2}{12} = \frac{1}{6} \qquad u'(0) = 0 \end{cases}$$

We now solve the IVP:

$$u(t) = u_H(t) + v(t),$$

$$u'' + 256\,u = 0$$

$$r^2 + 256 = 0, \quad r = \pm 16i$$

$$u_H(t) = c_1 \cos 16t + c_2 \sin 16t.$$

$$v(t) = A\cos 3t + B \sin 3t$$

$$v'(t) = -3A \sin 3t + 3B\cos 3t$$

$$v''(t) = -9A\cos 3t - 9B\sin 3t$$

Substitute:

$$v''(t) + 256\,v(t) = 16\cos 3t$$

$$-9A\cos 3t - 9B\sin 3t + 256A\cos 3t + 256B\sin 3t = 16\cos 3t$$

$$247A\cos 3t + 247B\sin 3t = 16\cos 3t$$

$$247A = 16 \qquad 247B = 0$$

$$A = \frac{16}{247} \qquad B = 0$$

$$\therefore \quad v(t) = \frac{16}{247}\cos 3t$$

$$\Rightarrow \quad u(t) = c_1\cos 16t + c_2 \sin 16t + \frac{16}{247}\cos 3t.$$

$$u(0) = c_1 + \frac{16}{247} = \frac{1}{6} \Rightarrow c_1 = \frac{1}{6} - \frac{16}{247} = \frac{151}{1482}$$

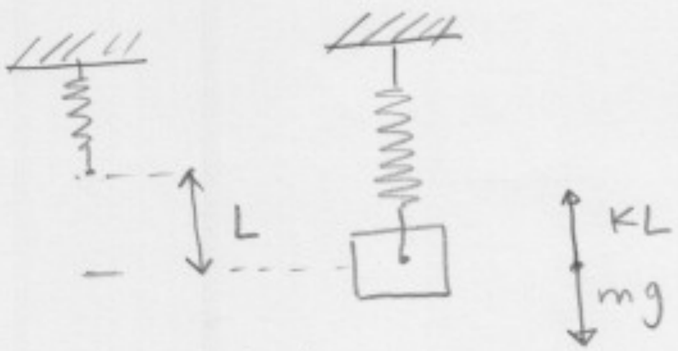
$$u'(t) = -16c_1\sin 16t + 16c_2\cos 16t - \frac{48}{247}\sin 3t$$

$$u'(0) = 16c_2 = 0 \Rightarrow c_2 = 0$$

Hence, the solution is:

$$\boxed{u(t) = \frac{151}{1482}\cos 16t + \frac{16}{247}\cos 3t}$$

<u>Ex</u>: A mass of 5 Kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin\frac{t}{2}$ Newtons and moves in a medium that imparts a viscous force of 2N when the speed of the mass is 4 cm/sec.
If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/sec, formulate the IVP describing the motion of the mass



$L = 0.1$ m

$mg = KL$ is Hooke's law

$mg = 5(9.8) = 49 \Rightarrow KL = 49$

$\Rightarrow K = \frac{49}{0.1} = 490$

We have

$$u(0)=0$$
$$u'(0)=0.03$$

The damping force is proportional to velocity:

$$F_d(t)=\gamma u'(t)$$

We have that $F_d=2$ when velocity is 0.04.

Hence:

$$2=\gamma(0.04)$$
$$\Rightarrow \gamma=\frac{2}{0.04}=\frac{200}{4}=50$$

Our initial value problem is:

$$\text{IVP}\begin{cases} 5u''+50u'+490=10\sin\frac{t}{2} \\ u(0)=0, \quad u'(0)=0.03 \end{cases}$$

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