Lesson 24

Ch 6.1: Definition of Laplace Transform

- Many practical engineering problems involve mechanical or electrical systems acted upon by discontinuous or impulsive forcing terms.
- For such problems the methods described in Chapter 3 are difficult to apply.
- In this chapter we use the Laplace transform to convert a problem for an unknown function f into a simpler problem for F, solve for F, and then recover f from its transform F.
- Given a known function $K(s,t)$, an **integral transform** of a function f is a relation of the form

$$F(s) = \int_{\alpha}^{\beta} K(s,t) f(t) dt, \quad \infty \le \alpha < \beta \le \infty$$

The Laplace Transform

- Let f be a function defined for $t \geq 0$, and satisfies certain conditions to be named later.
- The **Laplace Transform of f** is defined as

$$L\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st} f(t)\,dt$$

- Thus the kernel function is $K(s,t) = e^{-st}$.
- Since solutions of linear differential equations with constant coefficients are based on the exponential function, the Laplace transform is particularly useful for such equations.
- Note that the Laplace Transform is defined by an improper integral, and thus must be checked for convergence.
- On the next few slides, we review examples of improper integrals and piecewise continuous functions.

Example 1

- Consider the following improper integral.

$$\int_0^\infty e^{st}\,dt$$

- We can evaluate this integral as follows:

$$\int_0^\infty e^{st}\,dt = \lim_{b\to\infty}\int_0^b e^{st}\,dt = \lim_{b\to\infty}\left.\frac{e^{st}}{s}\right|_0^b = \frac{1}{s}\lim_{b\to\infty}\left(e^{sb}-1\right)$$

- Note that if $s = 0$, then $e^{st} = 1$. Thus the following two cases hold:

$$\int_0^\infty e^{st}\,dt = -\frac{1}{s}, \text{ if } s<0; \text{ and}$$

$$\int_0^\infty e^{st}\,dt \text{ diverges, if } s \geq 0.$$

Example 2

- Consider the following improper integral.

$$\int_0^\infty st \cos t\,dt$$

- We can evaluate this integral using integration by parts:

$$\int_0^\infty st \cos t\,dt = \lim_{b\to\infty} \int_0^b st \cos t\,dt$$

$u = st \quad dv = \cos t$
$du = s \quad v = \sin t$

$$= \lim_{b\to\infty} \left[st \sin t \Big|_0^b - \int_0^b s \sin t\,dt \right]$$

$$= \lim_{b\to\infty} \left[st \sin t \Big|_0^b + s \cos t \Big|_0^b \right]$$

$$= \lim_{b\to\infty} \left[sb \sin b + s(\cos b - 1) \right]$$

- Since this limit diverges, so does the original integral.

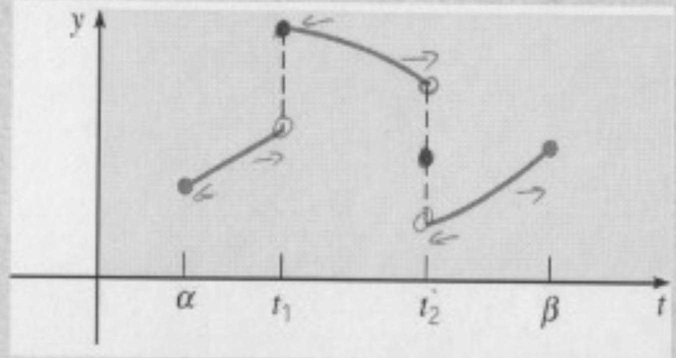
Piecewise Continuous Functions

- A function f is **piecewise continuous** on an interval $[a, b]$ if this interval can be partitioned by a finite number of points $a = t_0 < t_1 < \ldots < t_n = b$ such that

  (1) f is continuous on each (t_k, t_{k+1})

  (2) $\left|\lim_{t \to t_k^+} f(t)\right| < \infty, \quad k = 0, \ldots, n-1$

  (3) $\left|\lim_{t \to t_{k+1}^-} f(t)\right| < \infty, \quad k = 1, \ldots, n$



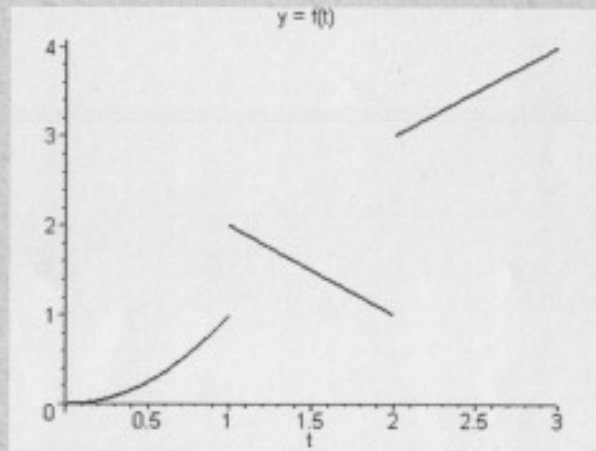
- In other words, f is piecewise continuous on $[a, b]$ if it is continuous there except for a finite number of jump discontinuities.

Example 3

- Consider the following piecewise-defined function f.

$$f(t) = \begin{cases} t^2, & 0 \le t \le 1 \\ 3 - t, & 1 < t \le 2 \\ t + 1 & 2 < t \le 3 \end{cases}$$

- From this definition of f, and from the graph of f below, we see that f is piecewise continuous on [0, 3].

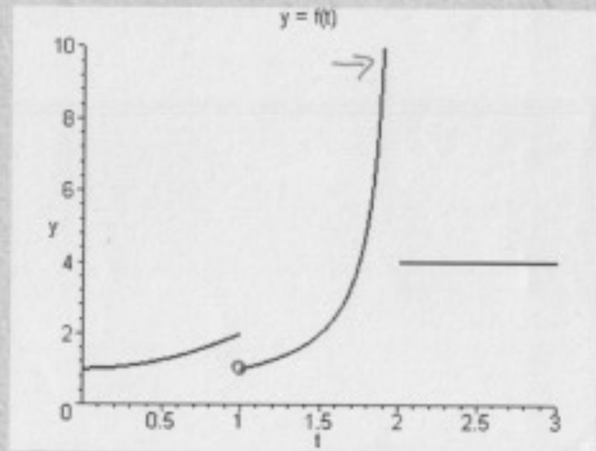


Example 4

- Consider the following piecewise-defined function f.

$$f(t) = \begin{cases} t^2+1, & 0 \le t \le 1 \\ (2-t)^{-1}, & 1 < t < 2 \\ 4, & 2 \le t \le 3 \end{cases}$$

- From this definition of f, and from the graph of f below, we see that f is not piecewise continuous on [0, 3].



Theorem 6.1.2

- Suppose that f is a function for which the following hold:
  - (1) f is piecewise continuous on $[0, b]$ for all $b > 0$.
  - (2) $|f(t)| \le Ke^{at}$ when $t \ge M$, for constants a, K, M, with $K, M > 0$.
- Then the Laplace Transform of f exists for $s > a$.

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \text{ finite}$$

- Note: A function f that satisfies the conditions specified above is said to to have **exponential order** as $t \to \infty$.

Example 5

※ Let $f(t) = 1$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$L\{1\} = \int_0^\infty e^{-st}\,dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-st}\,dt = \lim_{b\to\infty} \left[\frac{-1}{s} e^{-st} \right]_0^b$$

$$= -\lim_{b\to\infty} \frac{e^{-st}}{s}\bigg|_0^b = -\lim_{b\to\infty} \left(\frac{e^{-sb}}{s} - \frac{1}{s} \right)$$

$$= \frac{1}{s}, \quad s > 0$$

Example 6

※ Let $f(t) = e^{at}$ for $t \geq 0$. Then the Laplace transform $F(s)$ of f is:

$$L\{e^{at}\} = \int_0^{\infty} e^{-st} e^{at}\, dt$$

$$= \lim_{b\to\infty} \int_0^b e^{-(s-a)t}\, dt = \lim_{b\to\infty} \left[-\frac{1}{(s-a)} e^{-(s-a)t} \right]_0^b = -\lim_{b\to\infty} \left[\frac{e^{-(s-a)t}}{s-a} \right]_0^b$$

$$= -\lim_{b\to\infty} \frac{e^{-(s-a)t}}{s-a} \Bigg|_0^b = -\lim_{b\to\infty} \left[\frac{e^{-(s-a)b}}{s-a} - \frac{e^0}{s-a} \right]$$

$$= \frac{1}{s-a}, \quad s > a$$

Example 7

$\int u\,dv = uv] - \int v\,du$

※ Let $f(t) = \sin(at)$ for $t \geq 0$. Using integration by parts twice, the Laplace transform $F(s)$ of f is found as follows:

$u = e^{-st}$ $dv = \sin at$

$du = -se^{-st}$ $v = -\frac{1}{a}\cos at$

$$F(s) = L\{\sin(at)\} = \int_0^\infty e^{-st} \sin at\,dt = \lim_{b\to\infty} \int_0^b e^{-st} \sin at\,dt$$

$$= \lim_{b\to\infty}\left[-(e^{-st}\cos at)/a\Big|_0^b - \frac{s}{a}\int_0^b e^{-st}\cos at \right]$$

$$= \frac{1}{a} - \frac{s}{a}\lim_{b\to\infty}\left[\int_0^b e^{-st}\cos at\right]$$

$s > 0$

$u = e^{-st}$ $dv = \cos at$

$du = -se^{-st}$ $v = \frac{1}{a}\sin at$

$$= \frac{1}{a} - \frac{s}{a}\lim_{b\to\infty}\left[(e^{-st}\sin at)/a\Big|_0^b + \frac{s}{a}\int_0^b e^{-st}\sin at \right]$$

$$= \frac{1}{a} - \frac{s^2}{a^2}F(s) \Rightarrow F(s) = \frac{a}{s^2 + a^2}, \quad s > 0$$

Linearity of the Laplace Transform

- Suppose f and g are functions whose Laplace transforms exist for $s > a_1$ and $s > a_2$, respectively.
- Then, for s greater than the maximum of a_1 and a_2, the Laplace transform of $c_1 f(t) + c_2 g(t)$ exists. That is,

$$L\{c_1 f(t) + c_2 g(t)\} = \int_0^\infty e^{-st}\left[c_1 f(t) + c_2 g(t)\right]dt \quad \text{is finite}$$

with

$$\begin{aligned} L\{c_1 f(t) + c_2 g(t)\} &= c_1 \int_0^\infty e^{-st} f(t)\,dt + c_2 \int_0^\infty e^{-st} g(t)\,dt \\ &= c_1 L\{f(t)\} + c_2 L\{g(t)\} \end{aligned}$$

Example 8

- Let $f(t) = 5e^{-2t} - 3\sin(4t)$ for $t \geq 0$.
- Then by linearity of the Laplace transform, and using results of previous examples, the Laplace transform $F(s)$ of f is:

$$\begin{aligned} F(s) &= L\{f(t)\} \\ &= L\left\{5e^{-2t} - 3\sin(4t)\right\} \\ &= 5L\left\{e^{-2t}\right\} - 3L\left\{\sin(4t)\right\} \\ &= \frac{5}{s+2} - \frac{12}{s^2+16}, \quad s > 0 \end{aligned}$$