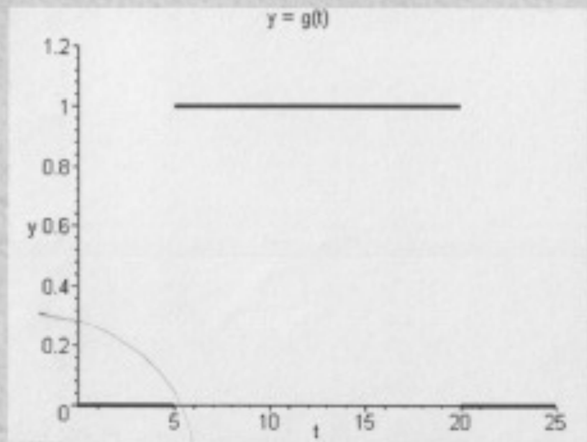
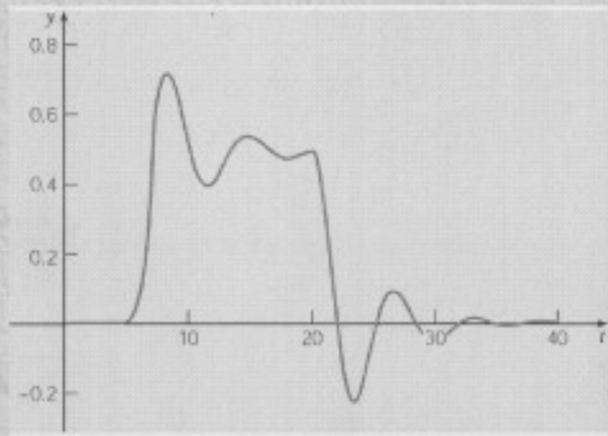
Lesson 27

Ch 6.4: Differential Equations with Discontinuous Forcing Functions

- In this section focus on examples of nonhomogeneous initial value problems in which the forcing function is discontinuous.

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \; y'(0) = y_0'$$





Example 1: Initial Value Problem (1 of 12)

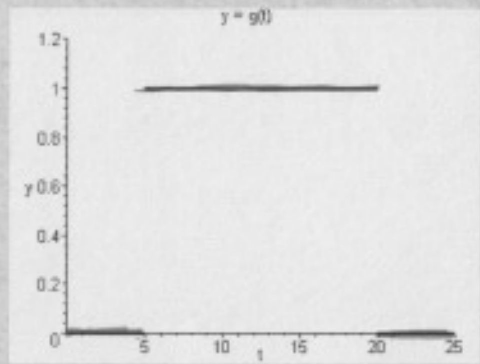
- Find the solution to the initial value problem

$$2y'' + y' + 2y = g(t), \quad y(0) = 0,\ y'(0) = 0$$

where

$$g(t) = u_5(t) - u_{20}(t) = \begin{cases} 1, & 5 \le t < 20 \\ 0, & 0 \le t < 5 \text{ and } t \ge 20 \end{cases}$$

- Such an initial value problem might model the response of a damped oscillator subject to $g(t)$, or current in a circuit for a unit voltage pulse.



$$2y'' + y' + 2y = u_5(t) - u_{20}(t), \quad y(0) = 0, \; y'(0) = 0$$

Example 1: Laplace Transform (2 of 12)

- Assume the conditions of Corollary 6.2.2 are met. Then

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{u_5(t)\} - L\{u_{20}(t)\}$$

or

$$\left[2s^2 L\{y\} - 2sy(0) - 2y'(0)\right] + \left[sL\{y\} - y(0)\right] + 2L\{y\} = \frac{e^{-5s} - e^{-20s}}{s}$$

- Letting $Y(s) = L\{y\}$,

$$\left(2s^2 + s + 2\right)Y(s) - (2s+1)y(0) - 2y'(0) = \left(e^{-5s} - e^{-20s}\right)/s$$

- Substituting in the initial conditions, we obtain

$$\left(2s^2 + s + 2\right)Y(s) = \left(e^{-5s} - e^{-20s}\right)/s$$

- Thus

$$Y(s) = \frac{\left(e^{-5s} - e^{-20s}\right)}{s\left(2s^2 + s + 2\right)} \qquad y(t) = \mathcal{L}^{-1}\left[\frac{e^{-5s} - e^{-20s}}{s(2s^2+s+2)}\right]$$

Example 1: Factoring $Y(s)$ (3 of 12)

- We have

$$Y(s) = \frac{\left(e^{-5s} - e^{-20s}\right)}{s\left(2s^2 + s + 2\right)} = \left(e^{-5s} - e^{-20s}\right)H(s) = e^{-5s}H(s) - e^{-20s}H(s)$$

where

$$H(s) = \frac{1}{s\left(2s^2 + s + 2\right)} \qquad \mathcal{L}(h(t)) = H(s).$$

- If we let $h(t) = L^{-1}\{H(s)\}$, then

$$y = \phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

by Theorem 6.3.1.

Example 1: Partial Fractions (4 of 12)

- Thus we examine $H(s)$, as follows.

$$H(s)=\frac{1}{s\left(2s^2+s+2\right)}=\frac{A}{s}+\frac{Bs+C}{2s^2+s+2}$$

- This partial fraction expansion yields the equations

$$(2A+B)s^2+(A+C)s+2A=1$$
$$\Rightarrow A=1/2,\ B=-1,\ C=-1/2$$

- Thus

$$H(s)=\frac{1/2}{s}-\frac{s+1/2}{2s^2+s+2}$$

Example 1: Completing the Square (5 of 12)

- Completing the square,

$$
\begin{aligned}
H(s) &= \frac{1/2}{s} - \frac{s+1/2}{2s^2+s+2} \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/2}{s^2+s/2+1}\right] \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/2}{s^2+s/2+1/16+15/16}\right] \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{s+1/2}{(s+1/4)^2+15/16}\right] \\
&= \frac{1/2}{s} - \frac{1}{2}\left[\frac{(s+1/4)+1/4}{(s+1/4)^2+15/16}\right]
\end{aligned}
$$

Example 1: Solution (6 of 12)

- Thus

$$H(s) = \frac{1/2}{s} - \frac{1}{2}\left[\frac{(s+1/4)+1/4}{(s+1/4)^2+15/16}\right]$$

$$= \frac{1/2}{s} - \frac{1}{2}\left[\frac{(s+1/4)}{(s+1/4)^2+15/16}\right] - \frac{1}{2\sqrt{15}}\left[\frac{\sqrt{15}/4}{(s+1/4)^2+15/16}\right]$$

and hence

$$h(t) = L^{-1}\{H(s)\} = \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos\left(\frac{\sqrt{15}}{4}t\right) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\frac{\sqrt{15}}{4}t\right)$$

- For $h(t)$ as given above, and recalling our previous results, the solution to the initial value problem is then

$$\phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20)$$

Example 1: Solution Graph (7 of 12)

- Thus the solution to the initial value problem is

$$\phi(t) = u_5(t)h(t-5) - u_{20}(t)h(t-20), \quad \text{where}$$

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-t/4}\cos\left(\sqrt{15}t/4\right) - \frac{1}{2\sqrt{15}}e^{-t/4}\sin\left(\sqrt{15}t/4\right)$$

- The graph of this solution is given below.

Example 2: Initial Value Problem

- Find the solution to the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 0,\ y'(0) = 0$$

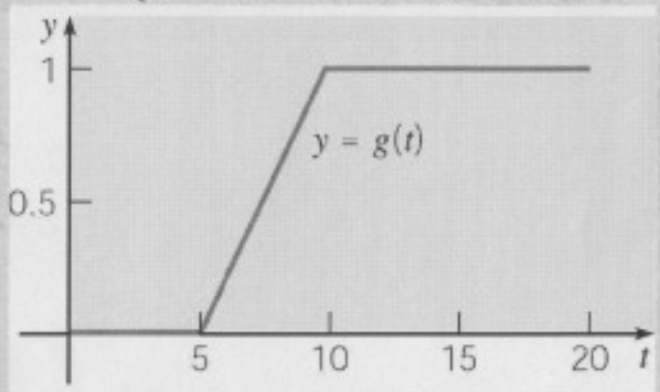
where

$$g(t) = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5} = \begin{cases} 0, & 0 \le t < 5 \\ (t-5)/5 & 5 \le t < 10 \\ 1, & t \ge 10 \end{cases}$$

$$g(t) = \frac{t-5}{5}\left(u_5(t) - u_{10}(t)\right) + u_{10}(t)$$

$$= u_5(t)\left(\frac{t-5}{5}\right) + u_{10}(t)\left[1 - \frac{t-5}{5}\right] = u_5(t)\left(\frac{t-5}{5}\right) - u_{10}(t)\left(\frac{t-10}{5}\right)$$

- The graph of forcing function $g(t)$ is given on right, and is known as ramp loading.



$$y'' + 4y = u_5(t)\frac{t-5}{5} - u_{10}(t)\frac{t-10}{5}, \quad y(0)=0,\ y'(0)=0$$

Example 2: Laplace Transform (2 of 12)

- Assume that this ODE has a solution $y = \phi(t)$ and that $\phi'(t)$ and $\phi''(t)$ satisfy the conditions of Corollary 6.2.2. Then

$$L\{y''\} + 4L\{y\} = \left[L\{u_5(t)(t-5)\}\right]/5 - \left[L\{u_{10}(t)(t-10)\}\right]/5$$

or

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + 4L\{y\} = \frac{e^{-5s} - e^{-10s}}{5s^2}$$

- Letting $Y(s) = L\{y\}$, and substituting in initial conditions,

$$\left(s^2+4\right)Y(s) = \left(e^{-5s} - e^{-10s}\right)/5s^2$$

- Thus

$$Y(s) = \frac{\left(e^{-5s} - e^{-10s}\right)}{5s^2\left(s^2+4\right)} = \frac{e^{-5s} - e^{-10s}}{5} H(s) = \frac{1}{5}\left[e^{-5s}H(s) - e^{-10s}H(s)\right]$$

Example 2: Factoring $Y(s)$ (3 of 12)

- We have

$$Y(s)=\frac{\left(e^{-5s}-e^{-10s}\right)}{5s^2\left(s^2+4\right)}=\frac{e^{-5s}-e^{-10s}}{5}H(s)$$

where

$$H(s)=\frac{1}{s^2\left(s^2+4\right)} \qquad \frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+4}$$

- If we let $h(t)=L^{-1}\{H(s)\}$, then

$$y=\phi(t)=\frac{1}{5}\left[u_5(t)h(t-5)-u_{10}(t)h(t-10)\right]$$

by Theorem 6.3.1.

Example 2: Partial Fractions (4 of 12)

- Thus we examine $H(s)$, as follows.

$$H(s) = \frac{1}{s^2\left(s^2+4\right)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+4}$$

- This partial fraction expansion yields the equations

$$(A+C)s^3 + (B+D)s^2 + 4As + 4B = 1$$
$$\Rightarrow A = 0,\ B = 1/4,\ C = 0,\ D = -1/4$$

- Thus

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$

Example 2: Solution (5 of 12)

- Thus

$$H(s) = \frac{1/4}{s^2} - \frac{1/4}{s^2+4}$$
$$= \frac{1}{4}\left[\frac{1}{s^2}\right] - \frac{1}{8}\left[\frac{2}{s^2+4}\right]$$

and hence

$$h(t) = L^{-1}\{H(s)\} = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- For $h(t)$ as given above, and recalling our previous results, the solution to the initial value problem is then

$$y = \phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right]$$

Example 2: Graph of Solution (6 of 12)

- Thus the solution to the initial value problem is

$$\phi(t) = \frac{1}{5}\left[u_5(t)h(t-5) - u_{10}(t)h(t-10)\right], \text{ where}$$

$$h(t) = \frac{1}{4}t - \frac{1}{8}\sin(2t)$$

- The graph of this solution is given below.