Lesson 28

Ch 6.5: Impulse Functions

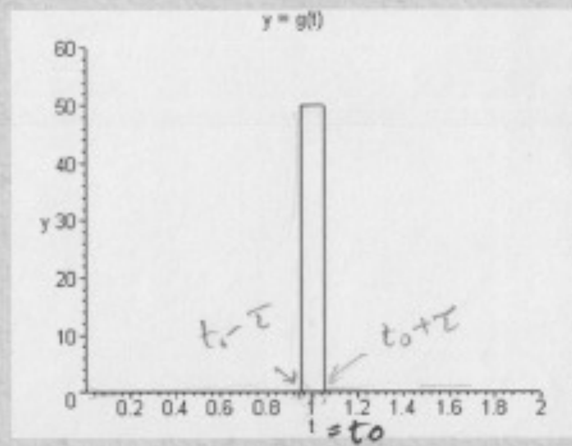
- In some applications, it is necessary to deal with phenomena of an impulsive nature.
- For example, an electrical circuit or mechanical system subject to a sudden voltage or force $g(t)$ of large magnitude that acts over a short time interval about t_0. The differential equation will then have the form

$$ay'' + by' + cy = g(t),$$

where

$$g(t) = \begin{cases} \text{big}, & t_0 - \tau < t < t_0 + \tau \\ 0, & \text{otherwise} \end{cases}$$

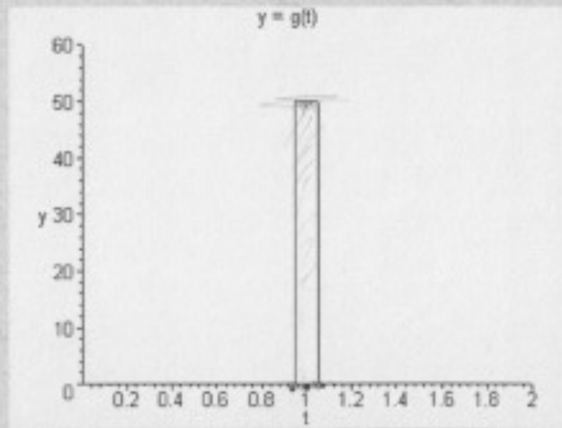
and $\tau > 0$ is small.



Measuring Impulse

- In a mechanical system, where $g(t)$ is a force, the total **impulse** of this force is measured by the integral

$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt = \int_{t_0-\tau}^{t_0+\tau} g(t)dt$$



- Note that if $g(t)$ has the form

$$g(t) = \begin{cases} c, & t_0 - \tau < t < t_0 + \tau \\ 0, & \text{otherwise} \end{cases}$$

then

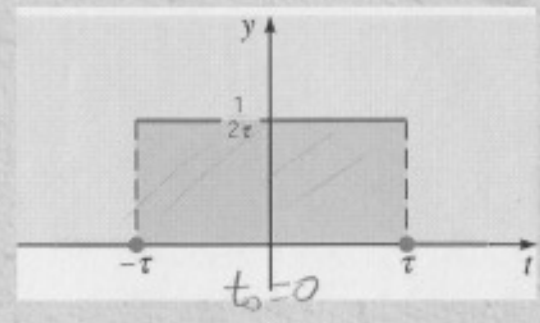
$$I(\tau) = \int_{-\infty}^{\infty} g(t)dt = \int_{t_0-\tau}^{t_0+\tau} g(t)dt = 2\tau c, \quad \tau > 0$$

- In particular, if $c = 1/(2\tau)$, then $I(\tau) = 1$ (independent of τ).

Unit Impulse Function

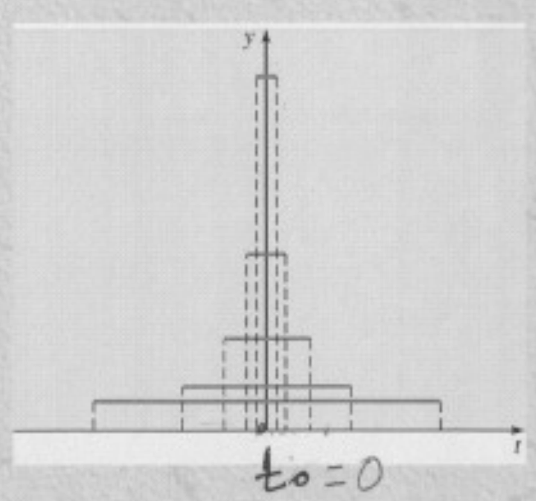
- Suppose the forcing function $d_\tau(t)$ has the form

$$d_\tau(t) = \begin{cases} 1/2\tau, & -\tau < t < \tau \\ 0, & \text{otherwise} \end{cases}$$



- Then as we have seen, $I(\tau) = 1$.
- We are interested $d_\tau(t)$ acting over shorter and shorter time intervals (i.e., $\tau \to 0$). See graph on right.
- Note that $d_\tau(t)$ gets taller and narrower as $\tau \to 0$. Thus for $t \neq 0$, we have

$$\lim_{\tau\to 0} d_\tau(t) = 0, \text{ and } \lim_{\tau\to 0} I(\tau) = 1$$



Dirac Delta Function

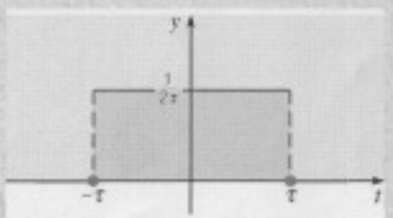
- Thus for $t \neq 0$, we have $\lim_{\tau \to 0} d_\tau(t) = 0$, and $\lim_{\tau \to 0} I(\tau) = 1$
- The **unit impulse function** δ is defined to have the properties

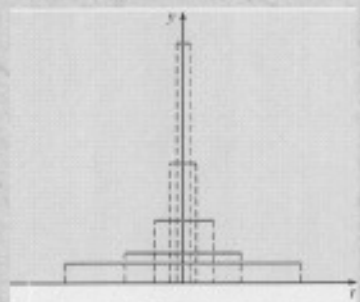
$$\delta(t) = 0 \text{ for } t \neq 0, \text{ and } \int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

$t_0 = 0$

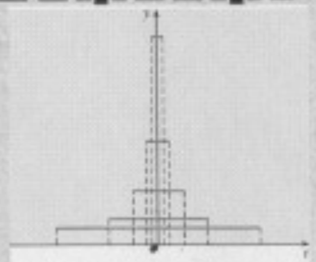
- The unit impulse function is an example of a generalized function and is usually called the **Dirac delta function**.
- In general, for a unit impulse at an arbitrary point t_0,

$$\delta(t - t_0) = 0 \text{ for } t \neq t_0, \text{ and } \int_{-\infty}^{\infty} \delta(t - t_0)\,dt = 1$$





Laplace Transform of δ (1 of 2)



- The Laplace Transform of δ is defined by

$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0} L\{d_\tau(t-t_0)\},\ t_0 > 0$$

and thus

$$L\{\delta(t-t_0)\} = \lim_{\tau\to 0}\int_0^\infty e^{-st} d_\tau(t-t_0)\,dt = \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} e^{-st}\,dt$$

$$= \lim_{\tau\to 0}\left.\frac{-e^{-st}}{2s\tau}\right|_{t_0-\tau}^{t_0+\tau} = \lim_{\tau\to 0}\frac{1}{2s\tau}\left[-e^{-s(t_0+\tau)} + e^{-s(t_0-\tau)}\right]$$

$$= \lim_{\tau\to 0}\frac{e^{-st_0}}{s\tau}\left[\frac{e^{s\tau}-e^{-s\tau}}{2}\right] = \cancel{e^{-st_0}\left[\lim_{\tau\to 0}\frac{\sinh(s\tau)}{s\tau}\right]} = \lim_{\tau\to 0}\left[\frac{e^{s\tau}-e^{-s\tau}}{\tau}\right]\frac{e^{-st_0}}{2s}$$

$$\cancel{= e^{-st_0}\left[\lim_{\tau\to 0}\frac{s\cosh(s\tau)}{s}\right]} = e^{-st_0} \quad = \frac{e^{-st_0}}{2s}\lim_{\tau\to 0}\frac{e^{s\tau}-e^{-s\tau}}{\tau}$$

$$= \frac{e^{-st_0}}{2s}\lim_{\tau\to 0}\frac{se^{s\tau}+se^{-s\tau}}{1} = \frac{e^{-st_0}}{2s}\cdot 2s = e^{-st_0}$$

Laplace Transform of δ (2 of 2)

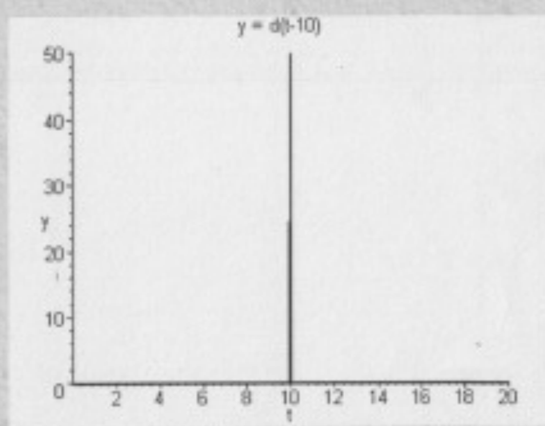
- Thus the Laplace Transform of δ is

$$L\{\delta(t-t_0)\} = e^{-st_0},\ t_0 > 0$$

- For Laplace Transform of δ at $t_0 = 0$, take limit as follows:

$$L\{\delta(t)\} = \lim_{t_0 \to 0} L\{d_\tau(t-t_0)\} = \lim_{\tau_0 \to 0} e^{-st_0} = 1$$

- For example, when $t_0 = 10$, we have $L\{\delta(t-10)\} = e^{-10s}$.



Product of Continuous Functions and δ

- The product of the delta function and a continuous function f can be integrated, using the mean value theorem for integrals:

$$\begin{aligned}\int_{-\infty}^{\infty} \delta(t-t_0) f(t)dt &= \lim_{\tau\to 0}\int_{-\infty}^{\infty} d_\tau(t-t_0) f(t)dt \\ &= \lim_{\tau\to 0}\frac{1}{2\tau}\int_{t_0-\tau}^{t_0+\tau} f(t)dt \\ &= \lim_{\tau\to 0}\frac{1}{2\tau}\left[2\tau\, f(t^*)\right] \quad (\text{where } t_0-\tau < t^* < t_0+\tau) \\ &= \lim_{\tau\to 0} f(t^*) \\ &= f(t_0)\end{aligned}$$

- Thus

$$\int_{-\infty}^{\infty} \delta(t-t_0) f(t)dt = f(t_0)$$

Example 1: Initial Value Problem (1 of 3)

- Consider the solution to the initial value problem

$$2y'' + y' + 2y = \delta(t-7), \quad y(0) = 0,\ y'(0) = 0$$

- Then

$$2L\{y''\} + L\{y'\} + 2L\{y\} = L\{\delta(t-7)\}$$

- Letting $Y(s) = L\{y\}$,

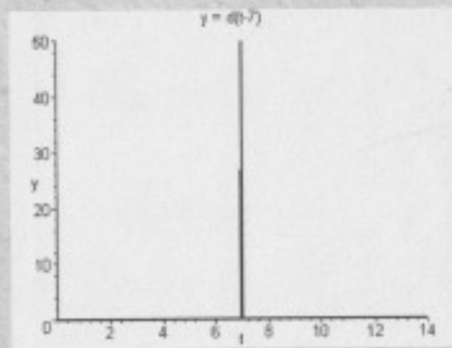
$$\left[2s^2Y(s) - 2s\cancel{y(0)}^{0} - 2\cancel{y'(0)}^{0}\right] + \left[sY(s) - \cancel{y(0)}^{0}\right] + 2Y(s) = e^{-7s}$$

- Substituting in the initial conditions, we obtain

$$\left(2s^2 + s + 2\right)Y(s) = e^{-7s}$$

or

$$Y(s) = \frac{e^{-7s}}{2s^2 + s + 2}$$



Example 1: Solution (2 of 3)

- We have

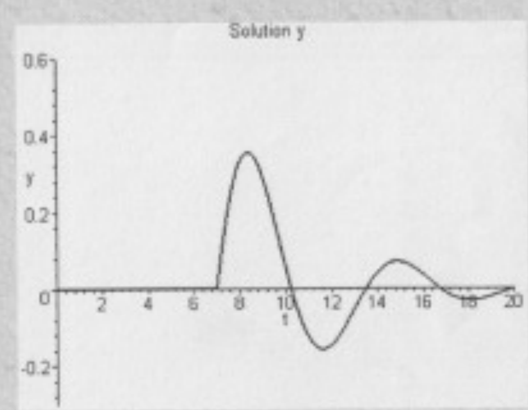
$$Y(s) = \frac{e^{-7s}}{2s^2 + s + 2}$$

- The partial fraction expansion of $Y(s)$ yields

$$Y(s) = \frac{4\, e^{-7s}}{2\sqrt{15}}\left[\frac{\sqrt{15}/4}{(s+1/4)^2 + 15/16}\right]$$

and hence

$$y(t) = \frac{2}{\sqrt{15}} u_7(t) e^{-(t-7)/4} \sin\frac{\sqrt{15}}{4}(t-7)$$



$2y'' + y' + 2y = \delta(t-7)$

Example 1: Solution Behavior (3 of 3)

- With homogeneous initial conditions at $t = 0$ and no external excitation until $t = 7$, there is no response on $(0, 7)$.
- The impulse at $t = 7$ produces a decaying oscillation that persists indefinitely.
- Response is continuous at $t = 7$ despite singularity in forcing function. Since y' has a jump discontinuity at $t = 7$, y'' has an infinite discontinuity there. Thus singularity in forcing function is balanced by a corresponding singularity in y''.