Lesson 29

Ch 6.6: The Convolution Integral

- Sometimes it is possible to write a Laplace transform $H(s)$ as $H(s) = F(s)G(s)$, where $F(s)$ and $G(s)$ are the transforms of known functions f and g, respectively.
- In this case we might expect $H(s)$ to be the transform of the product of f and g. That is, does

$$H(s) = F(s)G(s) = L\{f\}L\{g\} = L\{fg\}?$$

- On the next slide we give an example that shows that this equality does not hold, and hence the Laplace transform cannot in general be commuted with ordinary multiplication.
- In this section we examine the **convolution** of f and g, which can be viewed as a generalized product, and one for which the Laplace transform does commute.

Example 1

- Let $f(t) = 1$ and $g(t) = \sin(t)$. Recall that the Laplace Transforms of f and g are

$$L\{f(t)\} = L\{1\} = \frac{1}{s},\ L\{g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$$

- Thus

$$L\{f(t)g(t)\} = L\{\sin t\} = \frac{1}{s^2+1}$$

and

$$L\{f(t)\}L\{g(t)\} = \frac{1}{s(s^2+1)}$$

- Therefore for these functions it follows that

$$L\{f(t)g(t)\} \neq L\{f(t)\}L\{g(t)\}$$

Theorem 6.6.1

- Suppose $F(s) = L\{f(t)\}$ and $G(s) = L\{g(t)\}$ both exist for $s > a \geq 0$. Then $H(s) = F(s)G(s) = L\{h(t)\}$ for $s > a$, where

$$h(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t f(t)g(t-\tau)d\tau$$

- The function $h(t)$ is known as the **convolution** of f and g and the integrals above are known as **convolution integrals**.

- Note that the equality of the two convolution integrals can be seen by making the substitution $u = t - \tau$.
- The convolution integral defines a "generalized product" and can be written as $h(t) = (f*g)(t)$. See text for more details.

Proof of Theorem 6.6.1:

$$F(s)G(s) = \int_0^\infty e^{-su} f(u)\,du \int_0^\infty e^{-s\tau} g(\tau)\,d\tau$$

$$= \int_0^\infty \left[\int_0^\infty e^{-su} f(u)\,du\right] e^{-s\tau} g(\tau)\,d\tau$$

$$= \int_0^\infty \left[\int_0^\infty e^{-su} \cdot e^{-s\tau} f(u)\,du\right] g(\tau)\,d\tau$$

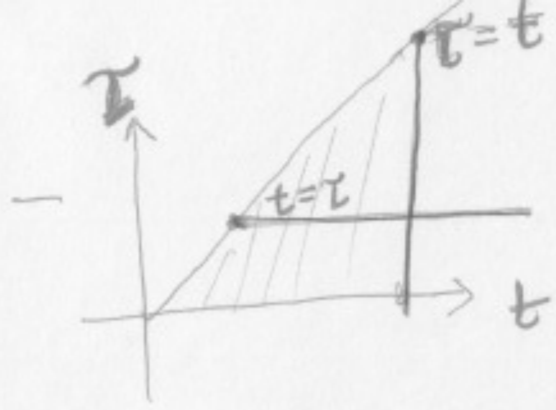
$$= \int_0^\infty \left[\int_0^\infty e^{-s(u+\tau)} f(u)\,du\right] g(\tau)\,d\tau$$

$t = u+\tau$ $\quad u \to t$

$dt = du$

$$= \int_0^\infty \left[\int_\tau^\infty e^{-st} f(t-\tau)\,dt\right] g(\tau)\,d\tau$$

$$= \int_0^\infty \int_\tau^\infty e^{-st} g(\tau) f(t-\tau)\,dt\,d\tau$$



$$= \int_0^\infty \int_0^t e^{-st} g(\tau) f(t-\tau)\, d\tau\, dt$$

$$= \int_0^\infty e^{-st} \left[\int_0^t f(t-\tau) g(\tau)\, d\tau \right] dt$$

$$= \mathcal{L}\{ (f*g)(t) \}.$$

Recal $(f*g)(t) = \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

Example 2

- Find the Laplace Transform of the function h given below.

$$h(t) = \int_0^t (t-\tau)\sin 2\tau \, d\tau \qquad (f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

- Solution: Note that $f(t) = t$ and $g(t) = \sin 2t$, with

$$F(s) = L\{f(t)\} = L\{t\} = \frac{1}{s^2}$$

$$G(s) = L\{g(t)\} = L\{\sin 2t\} = \frac{2}{s^2+4}$$

- Thus by Theorem 6.6.1,

$$L\{h(t)\} = H(s) = F(s)G(s) = \frac{2}{s^2\left(s^2+4\right)}$$

Example 3: Find Inverse Transform (1 of 2)

- Find the inverse Laplace Transform of $H(s)$, given below.

$$H(s) = \frac{2}{s^2(s-2)}$$

- Solution: Let $F(s) = 2/s^2$ and $G(s) = 1/(s-2)$, with

$$f(t) = L^{-1}\{F(s)\} = 2t$$
$$g(t) = L^{-1}\{G(s)\} = e^{2t}$$

- Thus by Theorem 6.6.1,

$$L^{-1}\{H(s)\} = h(t) = 2\int_0^t (t-\tau)e^{2\tau}\,d\tau \quad = \frac{1}{2}e^{2t} - t - \frac{1}{2}$$

$u = \tau \quad dv = 2e^{2\tau}d\tau$

$du = d\tau \quad v = e^{2\tau}$

$h = f * g(t)$

$$(f*g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^t 2(t-\tau)e^{2\tau}d\tau = \int_0^t 2te^{2\tau}d\tau - \int_0^t 2\tau e^{2\tau}d\tau$$

$$= t\int_0^t 2e^{2\tau}d\tau - \int_0^t 2\tau e^{2\tau}d\tau = t\,e^{2\tau}\Big]_0^t - \left[\tau e^{2\tau}\Big|_0^t - \int_0^t e^{2\tau}\right]$$

$$= te^{2t} - t - \left[te^{2t} - \left[\tfrac{1}{2}e^{2\tau}\right]_0^t\right] = te^{2t} - t - \left[te^{2t} - \tfrac{1}{2}e^{2t} + \tfrac{1}{2}\right]$$

Example 3: Solution $h(t)$ (2 of 2)

- We can integrate to simplify $h(t)$, as follows.

$$
\begin{aligned}
h(t) &= 2\int_0^t (t-\tau)e^{2\tau}\,d\tau = 2t\int_0^t e^{2\tau}\,d\tau - 2\int_0^t \tau e^{2\tau}\,d\tau \\
&= te^{2\tau}\Big|_0^t - \left[\tau e^{2\tau}\Big|_0^t - \int_0^t e^{2\tau}\,d\tau\right] \\
&= t\left[e^{2t}-1\right] - \left[te^{2t} - \frac{1}{2}\left[e^{2t}-1\right]\right] \\
&= te^{2t} - t - te^{2t} + \frac{1}{2}e^{2t} - \frac{1}{2} \\
&= \frac{1}{2}e^{2t} - t - \frac{1}{2}
\end{aligned}
$$

Example 4: Initial Value Problem (1 of 4)

- Find the solution to the initial value problem

$$y'' + 4y = g(t), \quad y(0) = 3,\ y'(0) = -1$$

- Solution:

$$L\{y''\} + 4L\{y\} = L\{g(t)\}$$

- or

$$\left[s^2 L\{y\} - sy(0) - y'(0)\right] + 4L\{y\} = G(s)$$

- Letting $Y(s) = L\{y\}$, and substituting in initial conditions,

$$\left(s^2 + 4\right)Y(s) = 3s - 1 + G(s)$$

- Thus

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}$$

Example 4: Solution (2 of 4)

- We have

$$Y(s) = \frac{3s-1}{s^2+4} + \frac{G(s)}{s^2+4}$$
$$= 3\left[\frac{s}{s^2+4}\right] - \frac{1}{2}\left[\frac{2}{s^2+4}\right] + \frac{1}{2}\left[\frac{2}{s^2+4}\right]G(s)$$

- Thus

$$y(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2}\int_0^t \sin 2(t-\tau)g(\tau)\,d\tau$$

- Note that if $g(t)$ is given, then the convolution integral can be evaluated.