Section 7.1

Introduction to Systems of first order linear equations.

A system of simultaneous first order ordinary differential equations has the general form:

$$\boxed{\begin{aligned} x_1' &= F_1(t, x_1, x_2, \dots, x_n) \\ x_2' &= F_2(t, x_1, x_2, \dots, x_n) \\ &\vdots \\ x_n' &= F_n(t, x_1, x_2, \dots, x_n) \end{aligned}} \qquad (*)$$

where each $x_k(t)$ is a function of t. If each F_k is a linear function of $x_1, \dots, x_n$ then the system of equations is said to be linear, otherwise it is nonlinear.

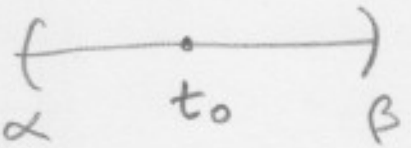
A system of first order ODE has a solution on the interval $I = (\alpha, \beta)$ if there exists n functions:

$$x_1 = \phi_1(t), \quad x_2 = \phi_2(t), \dots, \quad x_n = \phi_n(t)$$

that are differentiable on I and satisfy the system of equations at all points t in I.

The initial conditions may also be prescribed to give an IVP:

$$x_1(t_0) = x_1^0, \; x_2(t_0) = x_2^0, \; \ldots, \; x_n(t_0) = x_n^0$$



If each F_K is a <u>linear</u> function of $x_1, x_2, \ldots, x_n$, then the system of equations has the general form:

inear ystem (**)

$$\boxed{\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + g_1(t) \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + g_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + g_n(t) \end{aligned}}$$

If each of the $g_k(t)$ is zero on I, then the system is <u>homogeneous</u>, otherwise it is <u>non-homogeneous</u>.

Existence theorems

(For linear systems): Suppose each p_{ij}, $i=1,\dots,n$, $j=1,\dots,n$ and $g_1,\dots,g_n$ are continuous on an interval $I=(\alpha,\beta)$ with t_0 in I, and let $x_1^0, x_2^0, \dots, x_n^0$ be the initial conditions. Then there exists a unique solution to system (**):

$$x_1=\phi_1(t),\ x_2=\phi_2(t),\ \dots,\ x_n=\phi_n(t)$$

that satisfies the IVP, and exists throughout I.

(For non-linear system): Consider the non-linear system (*). Suppose $F_1,\dots,F_n$ and $\partial F_i/\partial x_j$, $i=1,\dots,n$, $j=1,\dots,n$, are continuous in the rectangle R of $(n+1)$-space defined by $\alpha<t<\beta$, $\alpha_1<x_1<\beta_1$, $\dots$, $\alpha_n<x_n<\beta_n$, and that the point $(t_0, x_1^0, x_2^0, \dots, x_n^0)$ is contained in R. Then, there exists h and a unique solution:

$$x_1=\phi_1(t),\ x_2=\phi_2(t),\ \dots,\ x_n=\phi_n(t)$$

to the non-linear system (*) defined for each t in the interval (t_0-h, t_0+h).

Ex: Consider the system;

$$x_1'(t) = 2x_1(t) - x_2(t) + e^t$$
$$x_2'(t) = 3x_1(t) - 2x_2(t) - e^t$$

This is a 2x2 linear system, non-homogeneous with constant coefficients (since all p_{ij} are constants).

The functions $x_1(t) = e^t + 2te^t$ and $x_2(t) = 2te^t$ solve the system. You can check this by computing $x_1'(t)$, $x_2'(t)$ and plugging directly in the above system. Another way is to work with matrix notation. Indeed, we can rewrite the system as:

$$\vec{x}'(t) = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x}(t) + \vec{g}(t),$$

with $\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$, $\vec{x}'(t) = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix}$ and $\vec{g}(t) = \begin{pmatrix} e^t \\ -e^t \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$,

We need to check that:

$$\vec{x}(t) = e^t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^t \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

is a solution. We compute:

$$\vec{x}'(t) = e^t\begin{pmatrix}1\\0\end{pmatrix} + (e^t + te^t)\begin{pmatrix}2\\2\end{pmatrix}$$

$$= e^t\begin{pmatrix}1\\0\end{pmatrix} + e^t\begin{pmatrix}2\\2\end{pmatrix} + te^t\begin{pmatrix}2\\2\end{pmatrix}$$

$$= \boxed{e^t\begin{pmatrix}3\\2\end{pmatrix} + te^t\begin{pmatrix}2\\2\end{pmatrix}}$$

On the other hand:

$$A\vec{x}(t) + \vec{g}(t) = \begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\left[e^t\begin{pmatrix}1\\0\end{pmatrix} + te^t\begin{pmatrix}2\\2\end{pmatrix}\right] + \begin{pmatrix}1\\-1\end{pmatrix}e^t$$

$$= e^t\begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + te^t\begin{pmatrix}2 & -1\\3 & -2\end{pmatrix}\begin{pmatrix}2\\2\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix}e^t$$

$$= e^t\begin{pmatrix}2\\3\end{pmatrix} + te^t\begin{pmatrix}2\\2\end{pmatrix} + \begin{pmatrix}1\\-1\end{pmatrix}e^t$$

$$= \boxed{\begin{pmatrix}3\\2\end{pmatrix}e^t + te^t\begin{pmatrix}2\\2\end{pmatrix}}$$

Since $\vec{x}'(t) = A\vec{x}(t) + \vec{g}(t)$ it follows that $\vec{x}(t)$ solves the system.

Ex: Consider the system:

$$x_1' = 3x_1 - 2x_2 \qquad x_1(0) = 3$$
$$x_2' = 2x_1 - 2x_2 \qquad x_2(0) = \tfrac{1}{2}$$

Convert this 2x2 system into a single second order equation.

From the first equation:

$$
\begin{aligned}
x_1''(t) &= 3x_1' - 2x_2' \\
&= 3x_1' - 2(2x_1 - 2x_2) \text{; using second equation} \\
&= 3x_1' - 4x_1 + 4x_2 \\
&= 3x_1' - 4x_1 + 4\left(\frac{3x_1 - x_1'}{2}\right) \text{; using first equation again.} \\
&= 3x_1' - 4x_1 + 6x_1 - 2x_1' \\
&= x_1' + 2x_1
\end{aligned}
$$

$$\Rightarrow \quad x_1'' - x_1' - 2x_1 = 0$$

$$x_1'(0) = 3x_1(0) - 2x_2(0) = 3(3) - 2\left(\tfrac{1}{2}\right) = 9 - 1 = 8$$

Hence, we can solve the system by solving the initial value problem:

$$\boxed{\begin{array}{l} x_1'' - x_1' - 2x_1 = 0 \\ x_1(0) = 3, \quad x_1'(0) = 8 \end{array}} \qquad (***)$$

Once we solve for $x_1(t)$ in (***) we can plug x_1 in the second equation to solve for $x_2(t)$.

<u>Ex</u>: The motion of a spring-mass system was described by the equation:

$$u''(t) + 16\,u'(t) + 192\,u(t) = 0$$

This second order equation can be converted into a system of first order equations by letting:

$$x_1 = u \quad , \quad x_2 = u' .$$

Indeed:

$$x_1'(t) = u'(t) = x_2(t)$$

$$u'' + 16u' + 192\,u = x_2'(t) + 16\,x_2(t) + 192\,x_1(t) = 0$$

$$x_1'(t) = x_2(t)$$
$$x_2'(t) = -192\,x_1 - 16\,x_2 ,$$

or in matrix notation:

$$\vec{x}\,'(t) = \begin{pmatrix} 0 & 1 \\ -192 & -16 \end{pmatrix} \vec{x}(t) .$$

<u>Ex</u>: $y'' + y = 0, \quad 0 < t < 2\pi$

This equation can be written as a system of first order equations by letting:

$$x_1 = y \quad \text{and} \quad x_2 = y'.$$

Hence:

$$x_1' = x_2$$
$$x_2' = y'' = -y = -x_1$$

$$\Rightarrow \quad \boxed{\begin{array}{l} x_1' = x_2 \\ x_2' = -x_1 \end{array}}$$

A solution to this system is

$$x_1(t) = \sin t$$
$$x_2(t) = \cos t, \quad 0 < t < 2\pi$$

which is a parametric description for the unit circle: