Sections 7.3

Eigenvalues and eigenvectors of a matrix. Systems of equations; case 1 (different and real eigenvalues)

We seek solutions of the equation:

$$\boxed{A\vec{x} = \lambda \vec{x}, \quad \vec{x} \neq 0,} \quad (*)$$

where λ is a number. A is a $n \times n$ matrix

That is, we need to find all the non-zero vectors $\vec{x}$ and all numbers λ such that:

$$A\vec{x} - \lambda \vec{x} = \vec{0}$$

or $\quad A\vec{x} - \lambda I \vec{x} = \vec{0} \quad$; that is:

$$\boxed{(A - \lambda I)\vec{x} = \vec{0}} \quad (**)$$

From linear algebra, equation $(**)$ has a non-zero solution if and only if:

$$\boxed{\det (A - \lambda I) = 0} \quad (***)$$

Indeed, with $B = A - \lambda I$, if $\det B \neq 0$, then B^{-1} exists, and hence the equation $B\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$ since:

$$B^{-1}(B\vec{x}) = B^{-1}\vec{0} \quad \Rightarrow \quad \vec{x} = \vec{0}.$$

The values of λ that satisfy (***) are called <u>eigenvalues</u> of the matrix A, and the non-zero vectors that solve (**); that is, (*), are called the eigenvectors corresponding to the eigenvalue λ.

<u>Ex</u>: Find the eigenvalues and eigenvectors of the matrix:

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}.$$

We solve first (***):

$$A - \lambda I = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-2-\lambda) + 4 = 0$$

$$-6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda_1 = 2 \quad \lambda_2 = -1$$

- For $\lambda_1 = 2$ we proceed to compute the eigenvectors:

We need to solve (**):

$$(A-2I)\vec{x} = \vec{0}$$

or

$$\begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 - x_2 = 0$$

$$4x_1 - 4x_2 = 0$$ (this is a multiple of first equation).

$$x_1 = x_2$$

The space of all eigenvectors corresponding to an eigenvalue λ is called the <u>eigenspace</u> for λ.

In our case, the eigenspace for $\lambda_1 = 2$ is:

$$\left\{ \begin{pmatrix} r \\ r \end{pmatrix} = r\begin{pmatrix} 1 \\ 1 \end{pmatrix} : r \text{ is any real number} \right\}$$

The eigenspace is always a vector space, and in this case is of dimension 1, generated by the eigenvector $\vec{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- We now compute the eigenvectors for $\lambda_2 = -1$:

$$\underbrace{\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}}_{(A-(-1)I)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(A-(-1)I)\vec{x} = \vec{0}$$

$$4x_1 - x_2 = 0$$
$$x_2 = 4x_1$$

The eigenspace corresponding to $\lambda_2 = -1$ is the one-dimensional vector space:

$$\left\{ \begin{pmatrix} r \\ 4r \end{pmatrix} = r\begin{pmatrix} 1 \\ 4 \end{pmatrix} : r \text{ is any real number} \right\},$$

generated by the eigenvector $\vec{e}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

We have found:

$$\lambda_1 = 2, \quad \vec{e}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \lambda_2 = -1, \quad \vec{e}_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

In particular, note that:

$$\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad A\vec{e}_1 = \lambda_1 \vec{e}_1$$

$$\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \end{pmatrix} = -\begin{pmatrix} 1 \\ 4 \end{pmatrix}, \quad A\vec{e}_2 = \lambda_2 \vec{e}_2$$

These equations are also true if we replace $\vec{e}_1$ and $\vec{e}_2$ by any other element of the corresponding eigenspace.

Ex: Find the general solution of the system of equations:

$$\vec{x}'(t) = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}(t)$$

or
$$x_1'(t) = 3x_1(t) - 2x_2(t)$$
$$x_2'(t) = 2x_1(t) - 2x_2(t)$$

We compute first the eigenvalues and eigenvectors. We then show how to use this information to obtain the general solution of the equation.

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-2-\lambda) + 4 = 0$$
$$-6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$
$$\lambda^2 - \lambda - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0$$
$$\lambda_1 = 2, \quad \lambda_2 = -1$$

- For $\lambda_1 = 2$

$$\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 - 2x_2 = 0$$
$$2x_1 - 4x_2 = 0$$
$$\Rightarrow x_1 = 2x_2$$

The eigenspace corresponding to $\lambda_1 = 2$ is:

$$\left\{ \begin{pmatrix} 2r \\ r \end{pmatrix} = r\begin{pmatrix} 2 \\ 1 \end{pmatrix} : r \text{ is any real number} \right\}.$$

For $\lambda_2 = -1$:

$$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{array}{l} 4x_1 - 2x_2 = 0 \\ 2x_1 - x_2 = 0 \end{array} \quad \Rightarrow \quad x_2 = 2x_1$$

the eigenspace corresponding to $\lambda_2 = -1$ is:

$$\left\{ \begin{pmatrix} r \\ 2r \end{pmatrix} = r\begin{pmatrix} 1 \\ 2 \end{pmatrix} : r \text{ is any real number} \right\}$$

We have:

$$\lambda_1 = 2, \quad \vec{e}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1, \quad \vec{e}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Note that:

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We consider the vectors:

$$\vec{x}^{(1)}(t) = e^{\lambda_1 t}\vec{e}_1 = e^{2t}\begin{pmatrix}2\\1\end{pmatrix}$$

$$\vec{x}^{(2)}(t) = e^{\lambda_2 t}\vec{e}_2 = e^{-t}\begin{pmatrix}1\\2\end{pmatrix}$$

$\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ are solutions to the system.

Indeed:

$$(\vec{x}^{(1)})'(t) = \lambda_1 e^{\lambda_1 t}\vec{e}_1 = e^{\lambda_1 t}\lambda_1\vec{e}_1$$

$$= e^{\lambda_1 t}A\vec{e}_1 \text{; since } A\vec{e}_1 = \lambda_1\vec{e}_1$$

$$= A(e^{\lambda_1 t}\vec{e}_1)$$

$$= A\vec{x}^{(1)}(t)$$

Hence: $\boxed{(\vec{x}^{(1)})'(t) = A\vec{x}^{(1)}(t).}$

Also:

$$(\vec{x}^{(2)})'(t) = \lambda_2 e^{\lambda_2 t}\vec{e}_2 = e^{\lambda_2 t}\lambda_2\vec{e}_2$$

$$= e^{\lambda_2 t}A\vec{e}_2 \text{; since } A\vec{e}_2 = \lambda_2\vec{e}_2$$

$$= A(e^{\lambda_2 t}\vec{e}_2)$$

$$= A\vec{x}^{(2)}(t).$$

Hence: $\boxed{(\vec{x}^{(2)})'(t) = A\vec{x}^{(2)}(t)}$

We check that $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$ are linearly independent solutions:

$$W(\vec{x}^{(1)}, \vec{x}^{(2)}) = \begin{vmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{vmatrix}$$

$$= 4e^{t} - e^{t} = 3e^{t} \neq 0 .$$

By the principle of superposition, any vector of the form:

$$c_1 \vec{x}^{(1)}(t) + c_2 \vec{x}^{(2)}(t),$$

is a solution to $\vec{x}'(t) = A\vec{x}(t)$. Since $W(\vec{x}^{(1)}, \vec{x}^{(2)}) \neq 0$, every solution is of this form.

That is, the space of solutions is a vector space of dimension 2 generated by $\vec{x}^{(1)}(t)$ and $\vec{x}^{(2)}(t)$.

We conclude that the general solution is:

$$\boxed{\begin{aligned} \vec{x}(t) &= c_1 e^{\lambda_1 t}\vec{e}_1 + c_2 e^{\lambda_2 t}\vec{e}_2 \\ \vec{x}(t) &= c_1 e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} \end{aligned}}$$

We can also write the solution as:

$$(1)\ \begin{cases} x_1(t) = 2c_1 e^{2t} + c_2 e^{-t} \\ x_2(t) = c_1 e^{2t} + 2c_2 e^{-t} \end{cases}$$

We have already checked that:

$$\vec{x}(t) = c_1 e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\2\end{pmatrix} \text{ is the}$$

general solution. We do it again:

$$\vec{x}\,'(t) = 2c_1 e^{2t}\begin{pmatrix}2\\1\end{pmatrix} - c_2 e^{-t}\begin{pmatrix}1\\2\end{pmatrix}$$

$$= c_1 e^{2t} \underbrace{2\begin{pmatrix}2\\1\end{pmatrix}}_{A\begin{pmatrix}2\\1\end{pmatrix}} + c_2 e^{-t} \underbrace{(-1)\begin{pmatrix}1\\2\end{pmatrix}}_{A\begin{pmatrix}1\\2\end{pmatrix}} \text{; since } A\vec{e}_1 = \lambda_1\vec{e}_1 \text{ and } A\vec{e}_2 = \lambda_2\vec{e}_2$$

$$= c_1 e^{2t} A\begin{pmatrix}2\\1\end{pmatrix} + c_2 e^{-t} A\begin{pmatrix}1\\2\end{pmatrix}$$

$$= c_1 A\, e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + c_2 A\, e^{-t}\begin{pmatrix}1\\2\end{pmatrix}$$

$$= A\left(c_1 e^{2t}\begin{pmatrix}2\\1\end{pmatrix} + c_2 e^{-t}\begin{pmatrix}1\\2\end{pmatrix}\right)$$

$$= A\vec{x}(t); \text{ hence } \vec{x}\,'(t) = A\vec{x}(t)$$

We can also check it is the solution by taking derivatives in (1) above and plug in the equation $\vec{x}\,'(t) = \begin{pmatrix}3&-2\\2&-2\end{pmatrix}\vec{x}(t)$. We have:

$$x_1'(t) = 4c_1 e^{2t} - c_2 e^{-t}$$
$$x_2'(t) = 2c_1 e^{2t} - 2c_2 e^{-t}$$

The equations are:

$$x_1'(t) = 3x_1(t) - 2x_2(t)$$
$$x_2'(t) = 2x_1(t) - 2x_2(t)$$

First equation:

$$x_1'(t) = 4c_1e^{2t} - c_2e^{-t} = 3(2c_1e^{2t} + c_2e^{-t}) - 2(c_1e^{2t} + 2c_2e^{-t})$$
$$= 6c_1e^{2t} + 3c_2e^{-t} - 2c_1e^{2t} - 4c_2e^{-t}$$
$$= 4c_1e^{2t} - c_2e^{-t} = 3x_1 - 2x_2 \checkmark$$

Second equation:

$$x_2'(t) = 2c_1e^{2t} - 2c_2e^{-t} = 2(2c_1e^{2t} + c_2e^{-t}) - 2(c_1e^{2t} + 2c_2e^{-t})$$
$$= 4c_1e^{2t} + 2c_2e^{-t} - 2c_1e^{2t} - 4c_2e^{-t}$$
$$= 2c_1e^{2t} - 2c_2e^{-t} = 2x_1 - 2x_2 \checkmark$$