Section 7.6
Complex eigenvalues

Find the general solution of the system:

$$\vec{x}'(t) = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \vec{x}(t)$$

or

$$x_1'(t) = 3x_1 - 2x_2$$
$$x_2'(t) = 4x_1 - x_2$$

We find eigenvalues and eigenvectors:

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix}$$

$$= (3-\lambda)(-1-\lambda) + 8 = 0$$
$$-3 - 3\lambda + \lambda + \lambda^2 + 8 = 0$$
$$\lambda^2 - 2\lambda + 5 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

We have two eigenvalues:

$\lambda_1 = 1+2i$, $\lambda_2 = 1-2i$

For $\lambda_1 = 1+2i$:

$$\begin{pmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 2-2i & -2 \\ 4 & -2-2i \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2(1-i)c_1 - 2c_2 = 0$$
$$4c_1 - 2(1+i)c_2 = 0$$

If we multiply the first equation times $(1+i)$:

$$2(1-i)(1+i)c_1 - 2(1+i)c_2 = 0$$
$$2(1-i^2)c_1 - 2(1+i)c_2 = 0$$
$$4c_1 - 2(1+i)c_2 = 0$$; which is the second equation.

We use first equation:

$$c_2 = (1-i)c_1$$

The eigenspace corresponding to $\lambda_1 = 1+2i$:

$$\left\{ \begin{pmatrix} c \\ (1-i)c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1-i \end{pmatrix} : c \text{ is any complex number} \right\}$$

For $\lambda_2 = 1-2i$,

$$\begin{pmatrix} 2+2i & -2 \\ 4 & -2+2i \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$2(1+i)c_1 - 2c_2 = 0$$
$$4c_1 - 2(1-i)c_2 = 0.$$

Again, if we multiply the first equation times $1-i$ we obtain the second equation.

$$2c_2 = 2(1+i)c_1$$
$$c_2 = (1+i)c_1$$

The eigenspace corresponding to $\lambda_2 = 1-2i$ is:

$$\left\{ \begin{pmatrix} c \\ (1+i)c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1+i \end{pmatrix} : c \text{ is any complex number} \right\}$$

We have found:

$$\lambda_1 = 1+2i, \quad \vec{e}_1 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$
$$\lambda_2 = 1-2i, \quad \vec{e}_2 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

In particular, notice that:

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1-i \end{pmatrix} = (1+2i)\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$
$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1+i \end{pmatrix} = (1-2i)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

We have found, two vector complex solutions.

We need two real solutions that are linearly independient.

The two complex solutions are:

$$\vec{x}^{(1)}(t) = e^{\lambda_1 t}\vec{e}_1 = e^{(1+2i)t}\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$\vec{x}^{(2)}(t) = e^{\lambda_2 t}\vec{e}_2 = e^{(1-2i)t}\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

We simplify using Euler's equations:

$$\vec{x}^{(1)}(t) = e^{t}\left(\cos 2t + i\sin 2t\right)\begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$= \begin{pmatrix} e^{t}\cos 2t + i e^{t}\sin 2t \\ e^{t}(1-i)\left(\cos 2t + i\sin 2t\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^{t}\cos 2t + i e^{t}\sin 2t \\ e^{t}\left(\cos 2t + i\sin 2t - i\cos 2t + \sin 2t\right) \end{pmatrix}$$

$$= \begin{pmatrix} e^{t}\cos 2t \\ e^{t}\cos 2t + e^{t}\sin 2t \end{pmatrix} + i\begin{pmatrix} e^{t}\sin 2t \\ e^{t}\sin 2t - e^{t}\cos 2t \end{pmatrix}$$

$$\vec{x}^{(2)}(t) = e^{t}(\cos 2t - i\sin 2t)\begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$= \begin{pmatrix} e^{t}\cos 2t - ie^{t}\sin 2t \\ e^{t}(1+i)(\cos 2t - i\sin 2t) \end{pmatrix}$$

$$= \begin{pmatrix} e^{t}\cos 2t - ie^{t}\sin 2t \\ e^{t}(\cos 2t - i\sin 2t + i\cos 2t + \sin 2t \end{pmatrix}$$

$$= \begin{pmatrix} e^{t}\cos 2t \\ e^{t}\cos 2t + e^{t}\sin 2t \end{pmatrix} - i\begin{pmatrix} e^{t}\sin 2t \\ e^{t}\sin 2t - e^{t}\cos 2t \end{pmatrix}$$

Using the principle of superposition we obtain:

$\vec{x}^{(3)}(t) = \vec{x}^{(1)}(t) + \vec{x}^{(2)}(t)$ is another solution

$$= 2\begin{pmatrix} e^{t}\cos 2t \\ e^{t}\cos 2t + e^{t}\sin 2t \end{pmatrix}$$

$\vec{x}^{(4)}(t) = \frac{1}{2}\,\vec{x}^{(3)}(t)$ is another solution

$$= \begin{pmatrix} e^{t}\cos 2t \\ e^{t}\cos 2t + e^{t}\sin 2t \end{pmatrix}$$

$\vec{x}^{(5)}(t) = \vec{x}^{(1)}(t) - \vec{x}^{(2)}(t)$ is another solution

$$= 2i \begin{pmatrix} e^t \sin 2t \\ e^t \sin 2t - e^t \cos 2t \end{pmatrix}$$

$\vec{x}^{(6)}(t) = \frac{1}{2i} \vec{x}^{(5)}(t)$ is also a solution

$$= \begin{pmatrix} e^t \sin 2t \\ e^t \sin 2t - e^t \cos 2t \end{pmatrix}$$

We have obtained $\vec{x}^{(4)}$ and $\vec{x}^{(6)}(t)$ two <u>real</u> solutions. They are linearly independent, since $W(\vec{x}^{(4)}(t), \vec{x}^{(6)}(t)) \neq 0$. Indeed:

$$W(\vec{x}^{(4)}(t), \vec{x}^{(6)}(t)) = \det \begin{pmatrix} e^t \cos 2t & e^t \sin 2t \\ e^t \sin 2t + e^t \cos 2t & e^t \sin 2t - e^t \cos 2t \end{pmatrix}$$

$$= \cancel{e^{2t} \sin 2t \cos 2t} - e^{2t} \cos^2 2t - e^{2t} \sin^2 2t \cancel{- e^{2t} \sin 2t \cos 2t}$$

$$= -e^{2t} \neq 0.$$

Hence, the general solution of the equation is:

$$\vec{x}(t) = c_1 \vec{x}^{(4)}(t) + c_2 \vec{x}^{(6)}(t)$$

$$= c_1 \begin{pmatrix} e^t \cos 2t \\ e^t \sin 2t + e^t \cos 2t \end{pmatrix} + c_2 \begin{pmatrix} e^t \sin 2t \\ e^t \sin 2t - e^t \cos 2t \end{pmatrix}$$

We can also write the solution as:

$$x_1(t) = c_1 e^t \cos 2t + c_2 e^t \sin 2t$$

$$x_2(t) = c_1 (e^t \sin 2t + e^t \cos 2t) + c_2 (e^t \sin 2t - e^t \cos 2t)$$

We now review the procedure for Case 2: λ_1 and λ_2 are complex eigenvalues.

In this case:

$$\lambda_1 = a+bi, \quad \lambda_2 = a-bi$$

$$\vec{e}_1 = \begin{pmatrix} \alpha+\beta i \\ \delta+\gamma i \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} \alpha-\beta i \\ \delta-\gamma i \end{pmatrix}$$

We use the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$ to simplify the complex solutions:

$$\vec{x}^{(1)}(t) = e^{\lambda_1 t}\vec{e}_1 \quad \text{and} \quad \vec{x}^{(2)}(t) = e^{\lambda_2 t}\vec{e}_2$$

We have:

$$e^{\lambda_1 t}\vec{e}_1 = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} + i\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

$$e^{\lambda_2 t}\vec{e}_2 = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} - i\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

Using the principle of superposition yields the two real solutions:

$$\vec{x}^{(1)}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \quad \vec{x}^{(2)}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}$$

The general solution is:

$$\vec{x}(t) = c_1\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} + c_2\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix}.$$

Remark: For 2×2 systems $\vec{x}'(t) = A\vec{x}(t)$ we need to prescribe the two initial conditions $x_1(0) = x_1^0$ and $x_2(0) = x_2^0$.

Ex: $$\vec{x}'(t) = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \vec{x}(t)$$
$$\vec{x}(0) = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

This system corresponds to case I: λ_1 and λ_2 are real and different. The general solution is (check it):

$$\vec{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$$

In order to compute c_1 and c_2 we impose the initial conditions:

$$\vec{x}(0) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\Rightarrow \left.\begin{array}{l} c_1 + c_2 = 1 \\ c_1 + 5c_2 = 3 \end{array}\right\} \Rightarrow \begin{array}{l} 4c_2 = 2 \Rightarrow c_2 = \frac{1}{2} \\ c_1 = 1 - c_2 = \frac{1}{2} \end{array}$$

Hence:

$$\vec{x}(t) = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ 5 \end{pmatrix} e^{3t}$$